Ch 2 Section 3

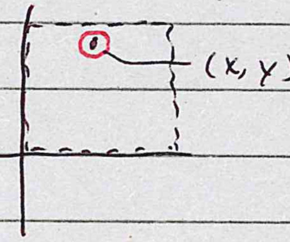
# Topology of Metric Spaces

**Def** Let $U \subset M$, a metric space. If $\forall p \in U$, $\exists\, \varepsilon > 0$ s.t. $B(p, \varepsilon) \subset U$, then $U$ is a open set. Equivalently, $U$ is open iff it is the union of open balls.

**Ex** In $\mathbb{R}$ the open balls are bdd open intervals. But we can show $(0,\infty)$ is open. Let $x \in (0,\infty)$. Then $x \in (0, 2x) \subset (0,\infty)$. Or, we could point out that

$$(0,\infty) = \bigcup_{n=1}^{\infty} (0, n).$$

**Ex** The open square $(0,1)\times(0,1) \subset \mathbb{R}^2$ is an open set.
Proof: Let $(x,y) \in (0,1)\times(0,1)$.
Let $r = \min\{x, 1-x, y, 1-y\}$.
Then $B((x,y), r/2) \subset (0,1)\times(0,1)$.
(Check that claim!)




But, is a square a union of open balls?
Yes,

$$(0,1)\times(0,1) = \bigcup_{(x,y)\in(0,1)\times(0,1)} B((x,y), r/2)$$

where $r$ is determined as above for each $(x,y)$.

**Def** Let $S$ be a subset of a metric space $M$. Let $x \in M$ ($x$ may or may not be in $S$). If $\exists$ an infinite seq $(s_n)$ is $S$ s.t. $s_n \to x$ then $x$ is a limit of $S$. Let

$$\lim S = \{x \in M \mid x \text{ is a limit of } S\}.$$

**Def** If $\lim S \subset S$, then $S$ is closed.

**Note** It is always the case that $S \subset \lim S$ since a seq can have repeated entries, that is if we let $s_n = s$ for all $n$, then $s_n \to s$.

**Ex** In $\mathbb{R}$ the set $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ is not closed, but the set $\{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ is closed.

In $\mathbb{R}$ $[a,b]$, $[a,\infty)$ and $(-\infty, b]$ are closed.

Notice that $\mathbb{R}$ be itself is closed and open!

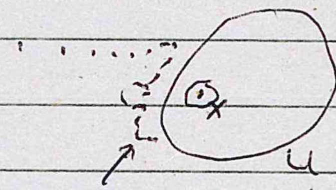
**Rmk** The textbook's terminology it not standard here. Most books define $x \in M$ to be a limit point of $S \subset M$ if $\exists$ $(s_n) \subset S$ s.t. $s_n \to x$ and $s_n \neq x$ $\forall n$. The set of limit pts is ~~called~~ denoted $S'$ and $S$ is closed if $S' \subset S$. But $S$ need be contained is $S'$. For example, $([0,1] \cup \{2\})' = [0,1]$. Limit points are sometimes called cluster pts or accumulation pts. See Section 6.

Thm (a) If $U$ is open in $M$, then $M-U$ is closed.

(b) If $C$ is closed in $M$, then $M-C$ is open.

The proofs are on the next page.

**Proof of (a)** Let $U \subset M$ be open. If $M-U$ is empty it is closed. Suppose $M-U \neq \phi$. Let $x_n \to x$ where each $x_n \in M-U$. Suppose $x \in U$. Then $\exists\, \varepsilon > 0$ s.t. $B(x,\varepsilon) \subset U$. But then $\forall n$, $x_n \notin B(x,\varepsilon)$, contradicting the definition of convergence. Thus $x \notin U$ and $M-U$ is closed.




It cannot get to x!

**Proof of (b)** Let $C \subset M$ be closed. If $M-C$ is empty it is open. Assume $M-C \neq \phi$. Let $x \in M-C$. Suppose, that $\forall\, \varepsilon > 0$ the open ball $B(x,\varepsilon)$ meets $C$. Then for each $n \in \mathbb{N}$ we can choose a point $x_n \in B(x, \frac{1}{n}) \cap C$. But then $x_n \to x$. But, since $C$ is closed, this means $x \in C$. This is a contradiction. Therefore, there is an $\varepsilon > 0$ s.t. $B(x,\varepsilon) \subset M-C$ and so $M-C$ is open.

But $x_n$ enters $M-C$!


"That's another fine contradiction you've gotten me into!"

**Def** The collection of all the open subsets of a metric space $M$ is called the topology of $M$.

**Thm** Let $\mathcal{T}$ be the topology of a metric space $M$. Then

(a) every union of members of $\mathcal{T}$ is in $\mathcal{T}$,

(b) the intersection of any finite subcollection from $\mathcal{T}$ is in $\mathcal{T}$,

(c) $\emptyset \in \mathcal{T}$ and $M \in \mathcal{T}$.

**Pf** The proof is easy. See textbook. Know this.

**Corollary** Dual statements are true for closed subsets of a metric space. That is,

(a) every intersection of closed sets is closed,

(b) every finite union of closed sets is closed,

(c) $M$ and $\emptyset$ are closed.

**Ex** This example shows the intersection of infinitely many open sets need not be open.

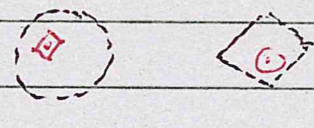
$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}.$$

**Ex** This example shows the union of infinitely many closed sets need not be closed.

$$\bigcup_{n=2}^{\infty} \left[\frac{1}{n}, 1\right] = (0, 1].$$

**Fact** Ealier we gave several different metrics for $\mathbb{R}^2$. They all give rise to the same topology.

**Idea of Proof:** We consider only the usual metric $d_2$ and the "taxi cab" metric $d_1((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| + |y_1 - y_2|$. The respective open balls look like:



Let $B_2$ be any open ball in the $d_2$ metric. Show that $\forall\, p \in B_2$, $\exists\, \varepsilon > 0$ s.t. $B_{d_1}(p, \varepsilon) \subset B_2$.

Let $B_1$ be any open ball in the $d_1$ metric. Show that $\forall\, p \in B_1$, $\exists\, \varepsilon > 0$ s.t. $B_{d_2}(p, \varepsilon) \subset B_1$.

These facts can be used with the properties of a topology to complete the proof.

**Def** Let $X$ be any set. Let $\mathcal{T}$ be a collection of subsets of $X$, s.t.

(a) every union of members of $\mathcal{T}$ is in $\mathcal{T}$,
(b) every intersection of finitely many members of $\mathcal{T}$ is in $\mathcal{T}$,
(c) $X \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$.

Then the pair $(X, \mathcal{T})$ is called a topological space.

There are entire courses on topology.