Still Section 3

Other Characterizations of Continuity

Thm (10) Let $f: M \to N$. The following are equivalent.

① $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon$.

② f^{-1} takes closed sets to closed sets.

③ f^{-1} takes open sets to open sets.

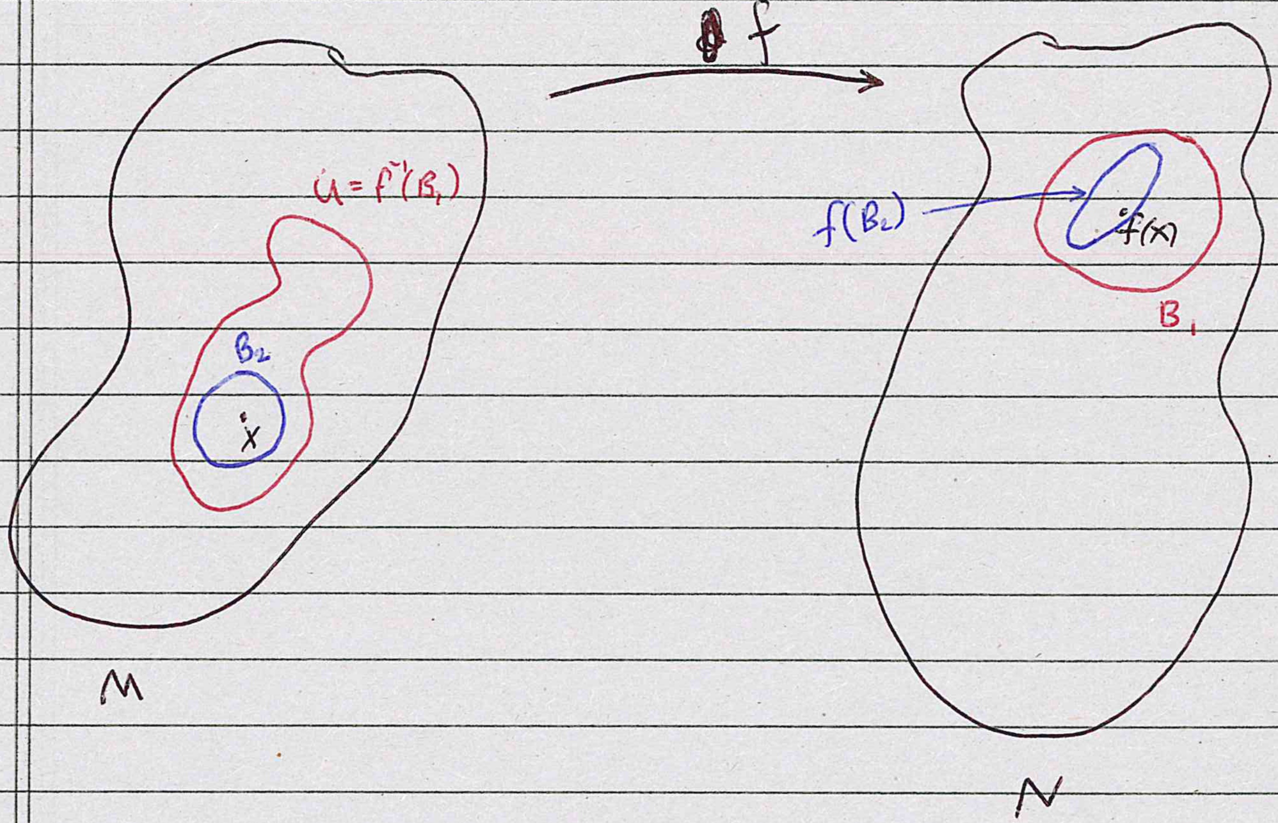
pf ① ⇒ ②. Assume ①. Let $C \subset N$ be closed. Let $D = f^{-1}(C) = \{x \in M \mid f(x) \in C\}$. To show that D is closed consider a seq (p_n), $p_n \in D$, with $p_n \to p$ in M. We will show $p \in D$.

We know $f(p_n) \to f(p)$. Since all $f(p_n) \in C$ and C is closed we have $f(p) \in C$. It follows that $p \in D$ since $p \in f^{-1}(f(p)) \subset D$. Thus D is closed.

② ⇒ ③. Assume ②. Let $U \subset N$ be open. $f^{-1}(U)$ is open is equivalent to $(f^{-1}(U))^c$ is closed. But,
$$(f^{-1}(U))^c = f^{-1}(U^c),$$
which is closed by assumption.

③ ⇒ ①. Assume ③. Let $\varepsilon > 0$. Let $x \in M$.
Let B_1 = the open ball $B(f(x), \varepsilon)$. Let $U = f^{-1}(B_1)$, which is open in M. Thus, $\exists\, \delta > 0$ s.t. $B(x, \delta) \subset U$. (Let $B_2 = B(x, \delta)$.) We now have

$$d_M(x, y) < \delta \iff y \in B_2 \Rightarrow f(y) \in B_1 \iff d_N(f(x), f(y)) < \varepsilon. \blacksquare$$



Study this drawing.