More Section 3

Closure, Interior and Boundary

Let S be a subset of a metric space M.

Def The closure of S is

$$cl(S) = \bar{S} = \bigcap K, \text{ where } S \subset K \text{ and } K \text{ is closed.}$$

Ex's $\overline{(0,1)} = [0,1]$. Let $A = \{\frac{1}{n} \mid n = 1,2,3\ldots\}$, $\bar{A} = \{0\} \cup A$.
$\bar{\mathbb{Q}} = \mathbb{R}$.

Fact $\bar{S} = \lim S$. See textbook for proof.

Def The interior of S is

$$int(S) = S^{o} = \bigcup U, \text{ where } U \subset S \text{ and } U \text{ is open.}$$

Ex's $int[0,1] = (0,1)$. $\mathbb{Q}^{o} = \phi$.

$$\left(\overline{[0,1) \cup (1,3)}\right)^{o} = (0,3).$$

Def The boundary of S is $\partial S = bd(S) = \bar{S} \cap \overline{S^c}$.
Some books call this the frontier of S because the term boundary means something different in the theory of manifolds.

Ex $\partial\,[0,1] = \{0,1\}$. $\partial\,\mathbb{Q} = \mathbb{R}$.

and Dense

Def Let M be a metric space (or any top. sp.) and let $D \subset M$. If $\overline{D} = M$, then we say D is dense in M.

Ex's $\mathbb{Q}$ is dense in $\mathbb{R}$. $(0,1)$ is dense in $[0,1]$.
$\{\frac{1}{n} \mid n \in \mathbb{N}\}$ is dense in $\{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$.

Thm An equivalent definition of dense is the following. D is dense in M iff $\forall p \in M$, every open nbhd of p meets D.

pf Suppose $\overline{D} = M$. Let $p \in M$. Suppose U is open with $p \in U \subset M \setminus D$. Then no seq. in D can converge to p. But now $p \notin \overline{D}$.

Suppose $\forall p \in M$ every open nbhd of p meets D. Let $p \in M$. Then $\forall n \in \mathbb{N}$, $B(p, \frac{1}{n}) \cap D \neq \emptyset$. Thus we can choose a point $x_n \in B(p, \frac{1}{n}) \cap D$ for each $n \in \mathbb{N}$. But now $x_n \to p$. Thus $p \in \overline{D}$ and we conclude that $\overline{D} = M$. $\square$

Rmk This proof is for metric spaces, but the thm is true for general top. sp's.