Yes, we are still in Section 3

Product Metrics

Def Suppose (M_1, d_1) and (M_2, d_2) are metric spaces. Let $M = M_1 \times M_2$. Then the Euclidean product metric for M is given by

$$d_E((a_1, a_2), (b_1, b_2)) = \sqrt{d_1(a_1, b_1)^2 + d_2(a_2, b_2)^2}$$

Two other useful metrics on M are

$$d_{max}((a_1, a_2), (b_1, b_2)) = \max\{d_1(a_1, b_1), d_2(a_2, b_2)\}$$

$$d_{sum}((a_1, a_2), (b_1, b_2)) = d_1(a_1, b_1) + d_2(a_2, b_2).$$

See exercise #38 for the proofs that these are metrics. (On $\mathbb{R}^2$ ~~~~ earlier I used d_∞ for d_{max} and d_1 for d_{sum}.)

Rmk Generalizing to finite products is straight foward. Dealing with infinite products is more involved.

See textbook for useful facts such as

$$d_{max} \leq d_E \leq d_{sum} \leq 2 d_{max}.$$

Completeness

Def Let (a_n) be a sequence in a metric space M. Then we say (a_n) is a Cauchy seq or that it satisfies the Cauchy condition if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. if $m, n > N$ then $d(a_m, a_n) < \varepsilon$.

Def A metric space is complete if all Cauchy seq's converge.

Rmk In any metric space convergent seq's must be Cauchy. We showed this for $\mathbb{R}$. For the general proof replace $|x-y|$ with $d(x,y)$. In a complete metric space we require that all Cauchy seq's do converge.

Ex $\mathbb{R}$, $\mathbb{R}^n$, $[0,1]$ are complete.
$\mathbb{Q}$, $(0,1)$, $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ are not.

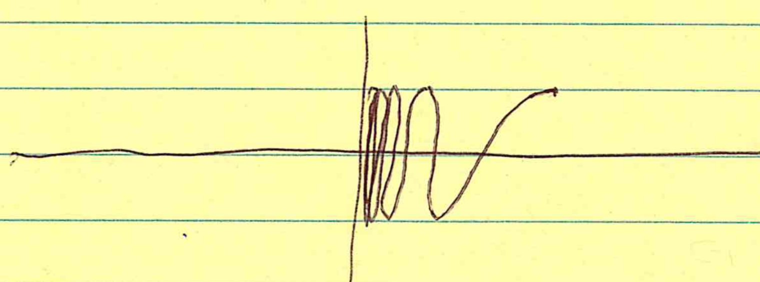
Thm Every closed subset of a complete metric space is complete in the subspace metric.

Pf: Easy. See textbook.

It is important to notice that completeness is not a topological property. For example $\mathbb{R}$ and $(-1,1)$ are homeomorphic (let $f:(-1,1)\to\mathbb{R}$ be $\tan(\frac{\pi}{2}x)$) but $\mathbb{R}$ is complete and $(-1,1)$ is not.

We saw that starting from $\mathbb{Q}$ we could add points a "complete it" to get $\mathbb{R}$. Given $(-1,1)$ we could add two pts to get $[-1,1]$, which is complete. Later, we will study a general method to form the completion of a metric space.

<u>Ex</u> Let $S=\{(x,y)\mid x\in(0,\frac{2}{\pi}]$ and $y=\sin(\frac{1}{x})\}$. Then S looks like this



What would the completion of S look like?

And now Section 3 is complete!!