Chapter 2
Section 4

Compactness

As the textbook says; "Compactness is the single most important concept in real analysis."

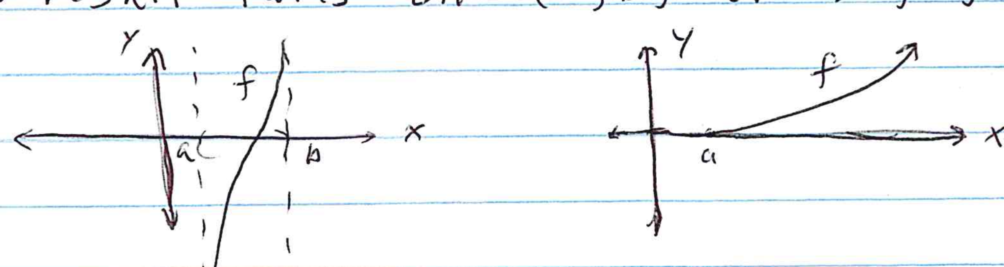
On the list of Supplement reading there is an article entitle "A pedagogical history of compactness," by Manya Raman-Sundstrom. Check it out if you have time.

There are several variants of the definition of compactness. The one we start with is not the standard def, but is more suited to our current aims. We will cover this other def. later.

Recall that the Extreme Value Thm of calculus takes place on a closed bounded interval:

Thm If $f:[a,b]\to\mathbb{R}$ is continuous, then $\exists$ values $p,q\in[a,b]$ s.t. $f(p)\le f(x)\le f(q)$ $\forall x\in[a,b]$.

This result fails on (a,b) or $[a,\infty)$:



The goal is to find the "right" generalized of being "closed and bounded" in a metric space. You think being closed and bdd would work straight away but there are problems with this.

Def A subset C of a metric space M is (sequentially) compact if every seq (a_n) in C has a convergent subseq. (a_{n_k}).

Rmk The empty set is considered compact and every finite set is obviously compact.

Rmk The def. makes sense even if we are in a topological space that does not have a metric.

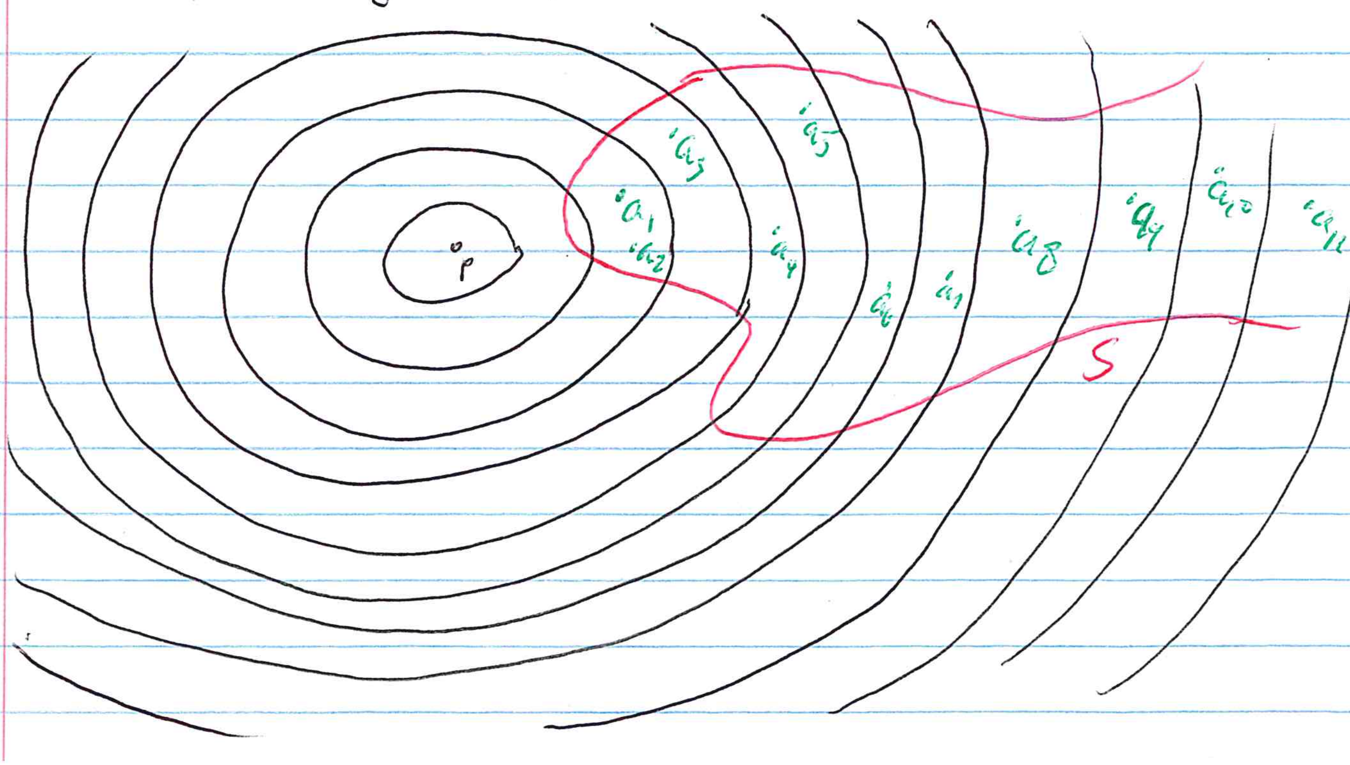
We establish some key properties.

Thm In a metric space every seq. comp. subset is closed and bdd.

pf (Know this!) Let $S \subset M$ be seq. comp. Let p be a limit of S and suppose $(a_n) \subset S$ converges to p, $a_n \to p$. We need to show $p \in S$. By the def we know $\exists$ a subseq (a_{n_k}) s.t. $a_{n_k} \to q$ where $q \in S$. But by Thm 1 on page 60 $a_{n_k} \to p$, and $p = q$. Thus S contains it limits and is closed.

Suppose S is not bounded. Let $p \in M$ and $B_n = B(p, n)$ $\forall n \in \mathbb{N}$. Since S is unbdd, $\forall n$ $\exists$ $a_n \in S \setminus B_n$. Now consider the seq. (a_n) in S.

But, as you can show, it does not have a convergent subseq. This contradicts the def. of seq. comp. Hence S must be bdd. $\square$



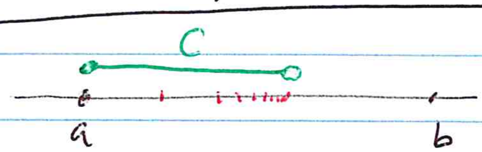
Thm Any closed, bdd interval, $[a,b] \subset \mathbb{R}$, is compact.

First, try to find a counterexample on your own. You cannot, but trying helps you to see **why** the thm has to be true.

Note that $(0,1]$ is not seq. comp. since $\{\frac{1}{n}\}$ has no subseq that has a lim in $(0,1]$. Likewise $[0,\infty)$ is not seq. comp. since $\{n\}$ has no conv. subseq. But, you should prove these claims.

pf This is tricky. ~~Let $C = \{x \in [a,b] |$~~ oops!
Let (x_n) be a seq. in $[a,b]$.

Let $C = \{x \in [a,b] \mid x_n < x$ for only finitely many values of $n\}$.

Example: 

Draw more examples before going on.

Clearly $a \in C$ since x_n is never $< a$. Thus $C \neq \emptyset$. Also, we know that b is an upper bound for C since $C \subset [a,b]$.

Let $c = \text{l.u.b. } C$.

Then $x_{n_k} \to c$, for some subseq. Suppose not.

Then $\exists\, \delta > 0$ s.t. at most finitely many x_n's are in $(c-\delta, c+\delta)$. (It can happen that finitely many $x_n = c$.) But then $c+\delta \in C$, contradicting the def of l.u.b. Thus $\exists$ a subseq (x_{n_k}) that converges to c. $\square$

Study this proof carefully. It uses a key idea that comes up often.

Thm Let M and N be metric spaces and suppose $A \subset M$ and $B \subset N$ are seq. comp. Then $A \times B$ is seq. comp. in the metric sp. $M \times N$ with the standard prod. metric.

pf: See textbook, pg 80.

By induction, the finite prod. of seq. comp. sets, $A_1 \ldots A_n$, is seq. comp.

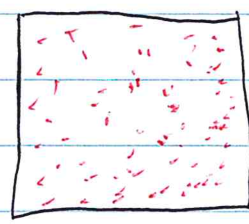
Thm The Bolzano-Weierstrass Thm

Every bdd seq in $\mathbb{R}^m$ has a conv. subseq.

Pf ~~Since~~ Let (x_n) be a bdd seq in $\mathbb{R}^m$. (Each x_n has m coordinates.) Since (x_n) is bdd it lives inside some box

$$B = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_m, b_m].$$

Since B is seq. comp. (x_n) has a conv. subseq. ∎



There must be at least one pt where the seq "clusters."

Thm A closed subset of a seq. comp. set is also seq. comp.

Pf Let $A \subset B \subset M$, where M is a metric space, B is compact and A is closed. Let (a_n) be a seq. in A. Then $\exists$ a subseq (a_{n_k}) and $b \in B$ s.t. $a_{n_k} \longrightarrow b$. Since b is a limit of A and A is closed, $b \in A$. ∎

Thm <u>Heine-Borel Thm</u>

Every closed bdd subset of $\mathbb{R}^m$ is seq. comp.

Pf Let $C \subset \mathbb{R}^m$ be closed and bdd. Since it is bdd it is inside a closed box, which is seq. compact. By the previous thm C is also seq. comp. ∎

Here is another improtant property of comp sets.

<u>Thm</u> Let M be a metric space. Let C_n, $n=1,2,3\ldots$ be a nested seq. of nonempty comp. sets. Nested means,

$$C_1 \supset C_2 \supset C_3 \supset C_4 \supset \cdots\cdots$$

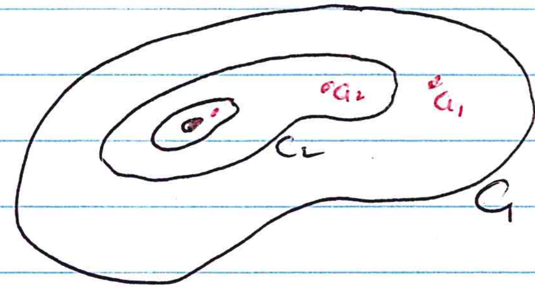
Then $\bigcap_{n=1}^{\infty} C_n \neq \phi$.

<u>Ex</u> $\bigcap_{n=1}^{\infty} [0, \frac{1}{n}] = \{0\}$. But $\bigcap_{n=1}^{\infty} (0, \frac{1}{n}] = \phi$ and

$$\bigcap_{n=1}^{\infty} [n, \infty) = \phi.$$

<u>Idea of Proof</u> $\forall n$ choose a pt $a_n \in C_n$.
Show (a_n) converges. Let $a_n \to p$.
Show $p \in C_n$ for every n.
Thus $p \in \bigcap C_n$.

See textbook, pg 82, for details.



Also study the corollary 35 in text.

Continuity and Compactness

Thm Let $f: M \to N$ be cont. and let $A \subset M$ be compact. Then $f(A) \subset N$ is compact.

Pf Suppose that (b_n) is a seq in $f(A)$.

$\forall n \in \mathbb{N}$ choose a point $a_n \in f^{-1}(\{b_n\})$.

$\exists$ a subseq (a_{n_k}) that converges to a pt in A; $a_{n_k} \to p \in A$.

But (b_{n_k}) is the cont. image of (a_{n_k}) under f. Thus $b_{n_k} \to f(p) \in f(A)$. ∎

Now we can prove a generalization of the extreme value thm.

Thm Let $f: M \to \mathbb{R}$ be cont. and let $S \subset M$ be seq. comp. Then $\exists\ p, q \in S$ s.t.

$$f(p) \leq f(x) \leq f(q) \quad \forall\, x \in S.$$

Pf $f(A) \subset \mathbb{R}$ is compact. Thus it is closed and bdd. Let $m = $ glb of $f(A)$ and $M = $ lub of $f(A)$. Since $f(A)$ is closed $m, M \in f(A)$. Thus $\exists\ p, q \in A$ s.t. $m = f(p)$ and $M = f(q)$. ∎

pretend this M is a different font.

Thm If M is compact and N is homeo. to M, then N is compact.

Pf Trivial.

This means that compactness is a topological property.

Recall that we defined $h: M \to N$ to be a homeo. if h is cont., one-to-one, onto and h^{-1} is cont. There are examples showing the first three properties do not imply h^{-1} is cont. But...

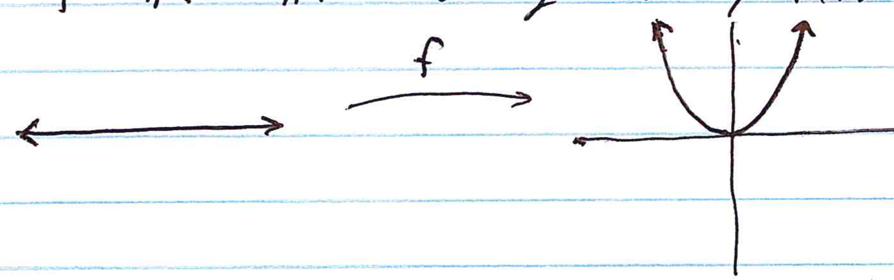
Thm If M is compact and $f: M \to N$ is a cont. bijection then f^{-1} is cont. and so f is a homeo.

Pf See textbook, pg 84–85.

Def Let $f: M \to N$ be cont., one-to-one with $f^{-1}: f(M) \to M$ cont. (so we are leaving out onto.) Then we say f is an embedding of M into N.

Note Many books use the word embedding to also mean f is differentiable. We have not defined this for metric spaces.

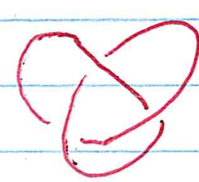
Ex Let $f: \mathbb{R} \to \mathbb{R}^2$ be given by $f(x) = (x, x^2)$.



Then f is an embedding of $\mathbb{R}$ into $\mathbb{R}^2$.

Obviously, if M is compact and $f: M \to N$ is an embedding, then $f(M)$ is compact.

Ex A knot in $\mathbb{R}^3$ is an embedding of the circle $\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$ into $\mathbb{R}^3$. We also call the image a knot. It follows that knots are compact.



A trefoil knot.

Uniform Continuity and Compactness

Uniform cont. is another key concept in real analysis.

Def $f: M \to N$ is uniformly continuous if

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. for all $p, q \in M$

$$d_M(p,q) < \delta \implies d_N(f(p), f(q)) < \varepsilon.$$

The difference we plain old cont. is that δ depends only on ε and is independent of p & q.

Thm Let $f: M \to N$ be cont. If M is compact, then f is also uniformly cont.

The proof is staight foward, but important.

Pf Suppose $f: M \to N$ is cont. M is compact, but that f is not unif. cont.

Then $\exists\, \varepsilon > 0$ s.t. $\forall\, \delta > 0$ $\exists\, p, q \in M$ s.t.

$$d_M(p,q) < \delta \quad \text{but} \quad d_N(f(p), f(q)) \geq \varepsilon.$$

(That is there is no $\delta > 0$ that works for all $p, q \in M$.)

Take $\delta = \frac{1}{n}$. Then for each $n \in \mathbb{N}$ choose points $p_n, q_n \in M$ s.t. $d_M(p_n, q_n) < \frac{1}{n}$ but $d_N(f(p_n), f(q_n)) \geq \varepsilon$.

Consider the sequences (p_n) and (q_n) in M.

Since M is comp. $\exists$ a subseq s.t. $p_{n_k} \to p \in M$.

You can use the Δ ineq to show $q_{n_k} \to p$.

Since f is cont. $f(p_{n_k}) \to f(p)$ and $f(q_{n_k}) \to f(p)$.

$\exists\, k_1$ s.t. $k \geq k_1 \Rightarrow d_N(f(p_{n_k}), f(p)) < \varepsilon/2$.

$\exists\, k_2$ s.t. $k \geq k_2 \Rightarrow d_N(f(q_{n_k}), f(p)) < \varepsilon/2$.

Then for $k \geq \max\{K_1, K_2\}$ we have

$$d(f(p_{n_k}), f(q_{n_k})) \leq d(f(p_{n_k}), f(p)) + d(f(q_{n_k}), f(p))$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This is a contradiction. Thus, f must be unif. cont. ☒