Ch 2 Section 5

Connectedness

Def Let M be a metric space. A separation of M is a pair U, V of disjoint nonempty open subsets of M whose union is M. That is,

$$U \neq \emptyset, V \neq \emptyset, U \cap V = \emptyset \text{ and } U \cup V = M.$$

If M does not have a separation, it is connected
If M does have a separation, it is disconnected.

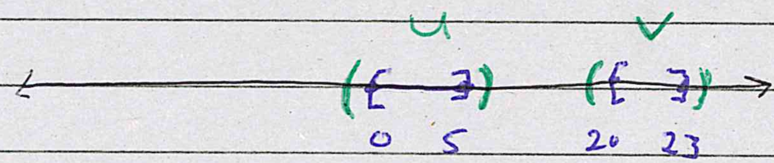
A similar definition applies to subsets of a metric space.

Def Let $S \subset M$, where M is a metric space. A separation of S is a pair U, V of open sets s.t.

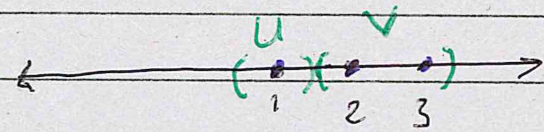
$$S \cap U \neq \emptyset, S \cap V \neq \emptyset, U \cap V = \emptyset \quad S \subset U \cup V.$$

If S does not have a separation, it is connected
If S does have a separation, it is disconnected.

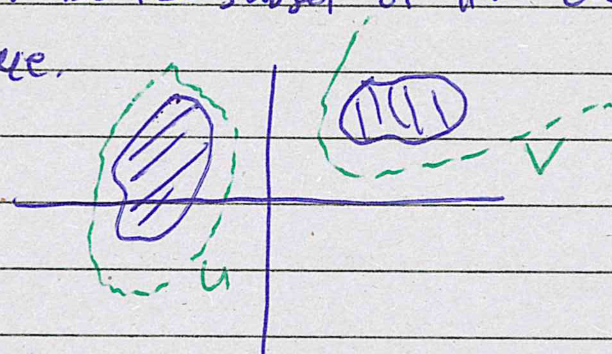
Examples ① Consider $S = [0,5] \cup [20,23] \subset \mathbb{R}$.
Let $U = (-\frac{1}{2}, 5\frac{1}{2})$ $V = (19\frac{1}{2}, 23\frac{1}{2})$,
Then U, V is a separation of S and so S is disconnected.



② Let $A = \{1, 2, 3\} \subset \mathbb{R}$. Let $U = (\frac{1}{2}, \frac{3}{2})$ and $V = (\frac{3}{2}, \frac{7}{2})$. Then U, V is a sep. of A and thus A is disconnected.



③ Let X be the subset of $\mathbb{R}^2$ depicted below in blue.



Then U, V, shown in green, is a separation.

The textbook uses a different definition that is equivalent.

<u>Book's def</u> Let M be a metric space. If M has a proper clopen subset A, then M is <u>disconnected</u>. The pair A, A^c is called a <u>separation</u> of M. If M is not disconnected, then we say it is <u>connected</u>.

Since A is proper, $A \neq M$ and $A \neq \phi$ by definition. Then $A^c = M - A$ is also proper. Clearly $A \cup A^c = M$. Since A is clopen, so is A^c. Thus the pair forms a sep. of M in the earlier sense.

Pugh's definition is clever, but the earlier definition ~~here~~ is more intuitive. It is given in Munkres' textbook, <u>Topology</u>.

<u>Exercise</u> Modify Pugh's def. to apply to subsets. He forgot to do this!

Connectedness is a topological property.

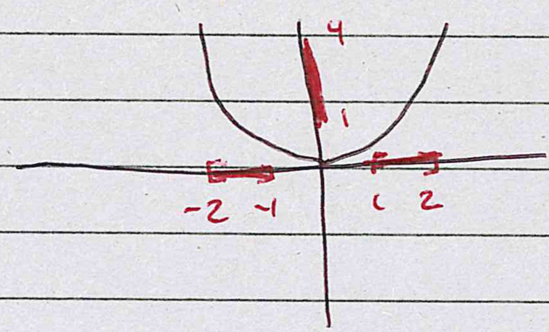
Thm Let M be connected. If $\exists\, f: M \to N$ that is cont. and onto, then N is connected.

Pf Suppose not and let U, V be a separation of N. Then $f^{-1}(U), f^{-1}(V)$ is a separation of M.

Cor Connectedness is a topological property. That is if M and N are homeomorphic either both are connected or both are disconnected.

Pf Let $h: M \to N$ be a homeo. If M is conn'd so is N. Since $h^{-1}: N \to M$ is also a homeo, if N is connected so is M.

Ex Note that the continuous image of a disconn'd ~~subset~~ space can be connected. Let $S = [-2,-1] \cup [1,2]$. Let $f(x) = x^2$. Then $f: S \to [1,4]$. An even simpiler example. Let $S = \{-1, 1\}$. Then $f: S \to \{1\}$.



Thm $\mathbb{R}$ is connected.

pf Suppose U, V is a separation of $\mathbb{R}$.
Then U is clopen (closed and open), not empty, and $U \neq \mathbb{R}$ (since $V \neq \phi$).

But U is a countable union of disjoint open intervals, whose end points are not in U.
But, U is closed, so it contains any end points. Therefore U has no end pts, that is $U = \mathbb{R}$. $\boxed{\text{L}}$

Note This proof uses material not in the 2nd edition. The proof in the 2nd ed. is different. Study it as well. Which do you like better?

Corollary The intervals in $\mathbb{R}$ are connected.

Partial Pf Open intervals, (a,b), $(-\infty,a)$ and (a,∞) are homeomorphic to $\mathbb{R}$. We give examples and let you fill in the details.

Let $f:\mathbb{R}\to(0,\infty)$ be $f(x)=e^x$. The inverse is $\ln x$. Modify this example to show $\mathbb{R}$ is homeo. to (a,∞) and $(-\infty,a)$ for any $a\in\mathbb{R}$.

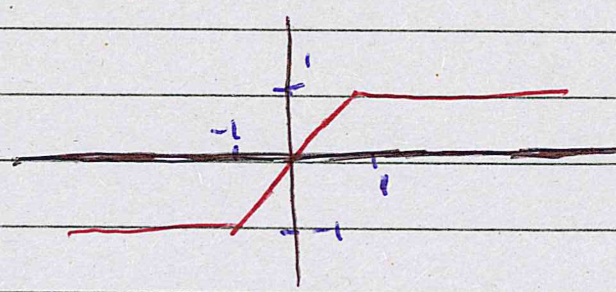
You have seen $\mathbb{R}$ is homeo to (a,b) before.

What about the closed intervals, $[a,b]$, $(-\infty,a]$ and $[a,\infty)$? They aren't homeo to $\mathbb{R}$, but they are the cont. image of $\mathbb{R}$.

Let $f:\mathbb{R}\to[-1,1]$ be given by

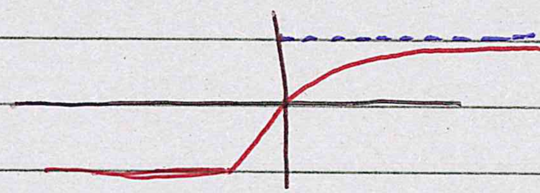
$$f(x)=\begin{cases}1 & x\geq 1\\ x & x\in[-1,1]\\ -1 & x\leq -1.\end{cases}$$

Here's the graph:



Of course I drew the graph first, and then cooked up a formula.

Now you come up with cont. function from $\mathbb{R}$ onto the closed intervals. All that ~~we~~ is left is to do the half open intervals $(a,b]$ and $[a,b)$. Here is a hint:



Go for it!

$\boxed{///}$

Ex Let $S^1 = \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2=1 \}$.

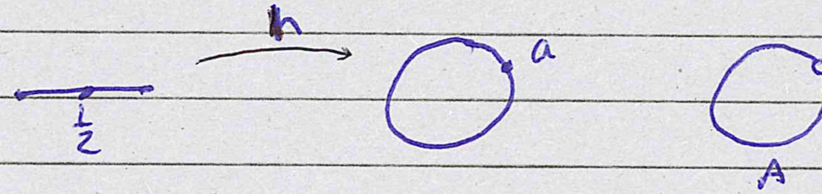
Let $f : \mathbb{R} \to S^1$ be given by

$$f(t) = (\sin t, \cos t).$$

Thus, S is a cont. image of $\mathbb{R}$ and so is connected.

Ex $[0,1]$ is not homeo. to S^1.

Pf Suppose $h : [0,1] \to S^1$ is a homeo.
Let $a = h(\frac{1}{2}) \in S^1$. Let $A = S^1 - \{a\}$.



You can show that A is connected since it is the image of an open interval. Now let h' be the restriction of h to $[0,\frac{1}{2}) \cup (\frac{1}{2}, 1]$. Then

$$h' : [0,\tfrac{1}{2}) \cup (\tfrac{1}{2}, 1] \to A$$

is a homeo. But this is impossible. Hence $h : [0,1] \to S^1$ cannot be a homeo. $\square$

Thm Let A and B be connected and suppose $A \cap B \neq \phi$. Then $A \cup B$ is connected.

Pf Suppose not. Assume U, V is a separation of $A \cup B$. Let $p \in A \cap B$. Then p is in U or V. Wlog assume $p \in U$. Since $p \in A$, $U \cap A \neq \phi$. If $A \not\subset U$, then $A \cap V \neq \phi$. But now U, V is a sep. of A. Thus $A \subset U$.

But, by the same reasoning $B \subset U$. Thus $V \cap (A \cup B) = \phi$. This is a contradiction! Thus there is no separation of $A \cup B$. ∎

Note The textbook shows that this result holds for unions of any number of connected sets with a point in common. It even holds for uncountable unions. Study the proof. It is on page 90.

As an application, the book uses this to show that the sphere

$$S^2 = \{(x,y,z) \mid x^2+y^2+z^2=1\}$$

is connected.

Problem Give an example showing that the intersection of conn'd sets need not be conn'd.

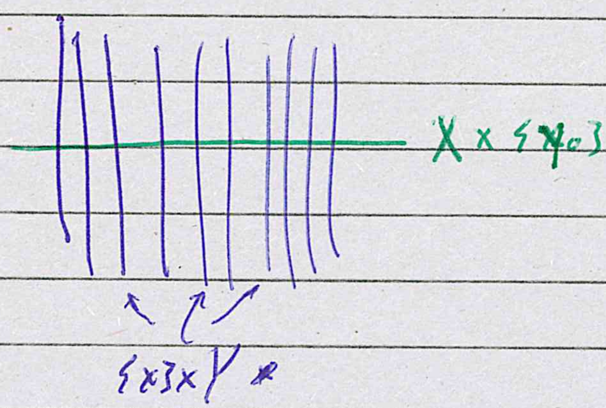
Thm The product of two connected spaces is conn'd.

pf Let X and Y be conn'd. We will show $X \times Y$ is conn'd using the result about unions of conn'd sets.

Let $x_0 \in X$. Then $\{x_0\} \times Y$ is homeo to Y and so is conn'd. Likewise $X \times \{y_0\}$, for $y_0 \in Y$, is conn'd. Thus the union $X \times \{y_0\} \cup \{x_0\} \times Y$ is conn'd since (x_0, y_0) is in their intersection.

Let $T_x = X \times \{y_0\} \cup \{x\} \times Y$. Each T_x is connected. The intersection $T_x \cap T_{x'}$ is $X \times \{y_0\}$.

Finally, $X \times Y = \bigcup_{x \in X} T_x$. ∎



Ex $\mathbb{R}^2$ is connected. The torus $S^1 \times S^1$ is conn'd.

Thm The Generalized I.V.T.

Let $f: M \to \mathbb{R}$ be continuous. Suppose M is a connected metric space and let $a, b \in M$, $a \neq b$. If $f(a) < y < f(b)$, then $\exists\, c \in M$ s.t. $f(c) = y$.

pf Suppose not. Let

$$U = \{ x \in M \mid f(x) < y \} \text{ and}$$

$$V = \{ x \in M \mid f(x) > y \}.$$

Now $a \in U$ and $b \in V$, so neither is empty. Clearly $U \cap V = \emptyset$. If $f(x)$ is never equal to y then $U \cup V = M$. But we still need to show U and V are open (the book skips this).

Notice $U = f^{-1}((-\infty, y))$ and $V = f^{-1}((y, \infty))$ are open since f is cont.

Therefore, M is not connected. This contradiction proves the Thm.

$\blacksquare$

Closure and Connectedness

Thm Let M be a metric space and $S \subset M$. If S is connected, then so is $\bar{S}$.

pf: Suppose $\bar{S}$ is disconnected. Let U, V be a separation of $\bar{S}$. Thus,

$$\bar{S} \cap U \neq \emptyset \quad \bar{S} \cap V \neq \emptyset, \quad U \cap V = \emptyset, \quad \bar{S} \subset U \cup V.$$

We claim that U, V is also a separation of S. Since $S \subset \bar{S} \subset U \cup V$ all we need to show is that $S \cap U \neq \emptyset$ and $S \cap V \neq \emptyset$.

Suppose $S \cap U = \emptyset$. Let $p \in \bar{S} - S$. But U is a nbhd of p that misses S. Since there is a seq. in S converging to p, this is impossible. Thus $S \cap U \neq \emptyset$. Likewise $S \cap V \neq \emptyset$.

Thus $\bar{S}$ disconn'd implies S is disconn'd.

Therefore S conn'd implies $\bar{S}$ is conn'd. $\blacksquare$

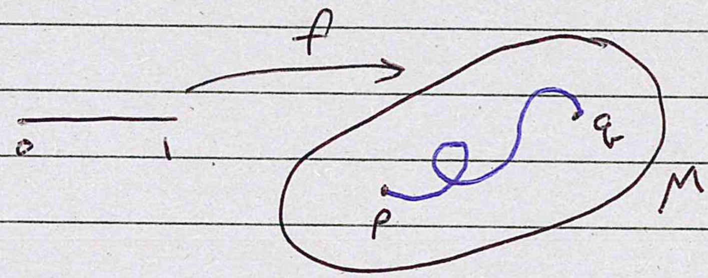
Ex Here we show S dis conn'd does not imply $\bar{S}$ is disconn'd.

$$\overline{[0, \tfrac{1}{2}) \cup (\tfrac{1}{2}, 1]} = [0, 1].$$

Here is another, $\bar{\mathbb{Q}} = \mathbb{R}$. To see that $\mathbb{Q}$ is disconn'd use the interiors of the Deke cut at $\sqrt{2}$.

Path Connectedness

Def Let M be a metric space with points p and q. A *path* from p to q is a continuous func. $f : [0,1] \to M$ s.t. $f(0) = p$ and $f(1) = q$.



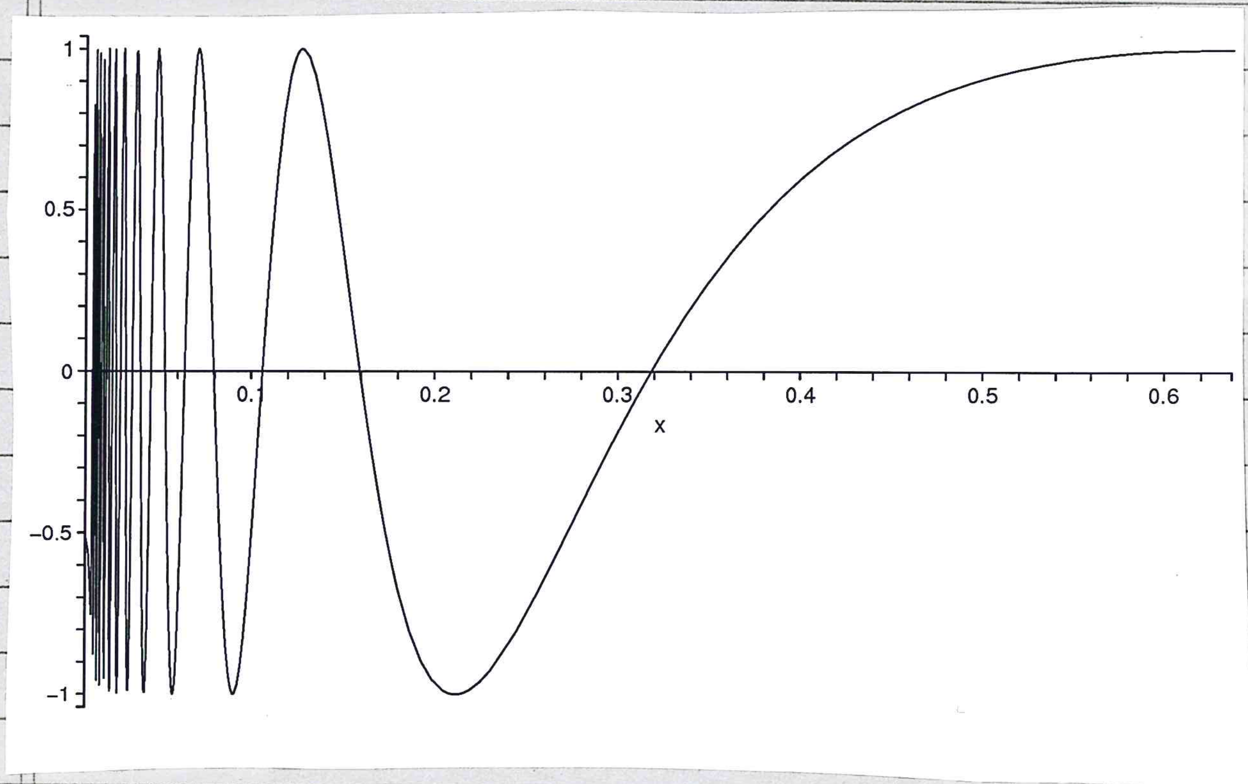
Def A subset S of a metric space is *path connected* if $\forall\, p, q \in S$ $\exists$ a path from p to q whose image is in S.

Thm A path conn'd space is conn'd.

Pf Suppose S is path conn'd but not conn'd. Let U, V be a separation of S. Let $p \in U \cap S$ and $q \in V \cap S$. Let $f : [0,1] \to S$ be a path from p to q. But then $f^{-1}(U), f^{-1}(V)$ is a separation of $[0,1]$, as you can check. $\blacksquare$

The Topologist's Sine curve

Let $S = \{(x,y) \in \mathbb{R}^2 \mid x \in (0, \frac{2}{\pi}] \text{ and } y = \sin(\frac{1}{x})\}$.



Since S can be regarded as the continuous image of $(0, \frac{2}{\pi}]$ it is connected. Thus $\bar{S}$ is conn'd. Note that

$$\bar{S} = S \cup \{0\} \times [-1, 1].$$

It can be shown that $\bar{S}$ is not path connected. See Munkres book Topology, page 157.

$\bar{S}$ is called the topologist's sine curve.

Here are two partial converses to the fact that path conn'd sets are conn'd.

Fact All connected subsets of $\mathbb{R}$ are also path conn'd. This follows by showing to conn'd subsets of $\mathbb{R}$ are intervals or a point.

Fact Let $U \subset \mathbb{R}^n$. If U is open and connected, then it is path conn'd. See exercise #64.