Ch 2 Section 6

Clustering and Condensing

Def Let M be a metric space, $S \subset M$ and $p \in M$.

p is a cluster point of S if $\forall \varepsilon > 0$ $B(p,\varepsilon) \cap S$ is infinite.

p is a condensation point of S if $\forall \varepsilon > 0$, $B(p,\varepsilon) \cap S$ is uncountable.

Recall p is a limit point of S if $\forall \varepsilon > 0$, $B(p,\varepsilon) \cap S \neq \emptyset$. (If $p_n \to p$ is a seq in S, then $p_n \in B(p,\varepsilon)$ for infinitely many n, but the seq could be all $p_n = p$, so the intersection can be finite.)

Ex For $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ $\frac{1}{10}$ is a limit pt, but not a cluster pt. 0 is a cluster pt. This set has no condensation points.

Every pt in $\mathbb{R}$ is a condensation pt. of $\mathbb{R}$
Every pt in $\mathbb{Q}$ is a cluster pt of $\mathbb{Q}$
Every pt in $\mathbb{Z}$ is a limit pt of $\mathbb{Z}$.

Warning Most books define limit points to be the same a cluster points. The term accumulation point is also used for this.

Thm The conditions below are equivalent to p being a cluster pt of S. (Any of these could be used as the def of a cl. pt. and some are used in other textbooks.)

(i) $\exists$ a seq of distinct points (no repeats) in S that converges to p.

(ii) $\forall$ open nbhd of p contains infinitely many points in S.

(iii) $\forall$ open nbhd of p contains at least two pts in S. (One could be p itself)

(iv) $\forall$ open nbhd of p contains at least one pt in S that is not p.

That (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) is obvious. See the textbook for the proof that (iv) $\Rightarrow$ (i). Pg 93.

Def Let S' denote the set of cluster pts of S.

Facts (a) $\bar{S} = S \cup S'$

(b) S is closed iff $S' \subset S$.

Try to prove these before reading the text books proofs on pg 93.

$\boxed{\text{Perfect Metric Spaces}}$

Def A metric space M is perfect if $M = M'$.

Ex $[0,1] \cup \{2\}$ is not perfect.

$\mathbb{Q}$, regarded as a metric space on its own, is prefect. But, if we ~~make an em~~ define prefect subsets of metric spaces in the obvious way, $\mathbb{Q}$ is not a perfect subset of $\mathbb{R}$.

$\mathbb{N}$ is not perfect on its own or as a subset of $\mathbb{R}$.

Obviously, perfect sets are closed.

The following thm is interesting, in part, because gives a conclusion about cardnality from only topological assumptions.

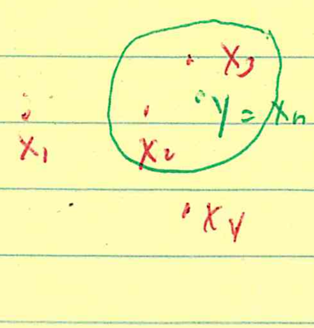
Thm. Every nonempty, perfect, complete metric space is uncountable.

Pf Suppose M is a counterexample. M cannot be finite, thus M is countably infinite. Let

$$M = \{x_1, x_2, x_3, \dots\}$$

be an enumeration of all the points of M. We will show $\exists\, y \in M$ s.t. $y \neq x_n \;\forall n$. This contradiction then proves the thm. Compare this to the proof that $\mathbb{R}$ is uncountable on page 32. We will construct a seq in M whose limit is y.

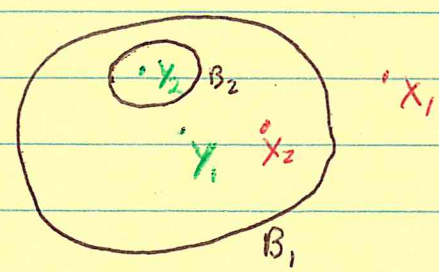
Let $y_1 \in \{x_2, x_3, x_4, \dots\}$. Choose a radius $r_1 \in (0,1)$ s.t. the closed ball $\overline{B(y_1, r_1)}$ does not contain x_1. All we need is for r_1 to be smaller than half the distance between x_1 and y_1. Let $B_1 = \overline{B(y_1, r_1)}$.



Pick a pt $y_2 \in \operatorname{int} B_1$ with $y_2 \neq x_2$. We can do this since y_2 is a cluster pt and the interior of B_1 is a nbhd of y_2, and so contain ∞ many pts.

Now choose $r_2 \in (0, \frac{1}{2})$ s.t. $B_2 = \overline{B(y_2, r_2)}$ does not contain x_2, and so that $B_2 \subset B_1$.

The second condition can always be met since y_2 is in the interior of B_1.



Note, $y_1 \in B_2$ is allowed.

We can repeat this process. Each time $r_n \in (0, \frac{1}{n})$ $B_n \subset B_{n-1}$ and B_n does not contain $\{x_1, x_2 \ldots x_n\}$.

Since $r_n \to 0$ the sequence (y_n) is Cauchy. (prove this.) Since M is complete $\exists y \in M$ s.t.

$$y_n \to y.$$

But now we have a problem. Since B_1 is closed and the seq $(y_n)_{n=1}^{\infty}$ is in B_1, $y \in B_1$.

Since B_2 is closed and the seq $(y_n)_{n=2}^{\infty}$ is in B_2, $y \in B_2$.

In fact $y \in B_n$, $\forall n$. Therefore $y \neq x_n$ $\forall n$. $\blacksquare$

For more on this topic see *Perfect subsets of the real line* by Jan Reimann at the link below.

http://www.personal.psu.edu/jsr25/Spring_11/574_Sp11_Syllabus.html

Continuinty of Arithmetic in $\mathbb{R}$

The basic operations of arithmetic can be show to be continuous. The proofs are straight forward. See the textbook.

Show that the vector space operations on $\mathbb{R}^n$ are also cont. That is

$$\begin{aligned} \oplus &: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n, \cancel{\text{and}} \\ \odot &: \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n, \text{ and} \\ \cdot &: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}, \end{aligned}$$

are cont.

Section 6 also covers closure, interior, boundary and boundedness, but we did these earlier.