§7 Covering Compactness

Def Let M be a metric space and $A \subset M$. Suppose that $\mathcal{U}$ is a collection of subsets of M. If A is in the union of all $U \in \mathcal{U}$, then we say $\mathcal{U}$ is a <u>covering</u> of A. Another way of saying this is that $\forall p \in A$, $\exists U \in \mathcal{U}$ s.t. $p \in U$.

If $\mathcal{V} \subset \mathcal{U}$, that all sets in $\mathcal{V}$ are also in $\mathcal{U}$, and $\mathcal{V}$ covers A we say $\mathcal{V}$ is a <u>subcovering</u> of A or that $\mathcal{U}$ <u>reduces</u> to $\mathcal{V}$.

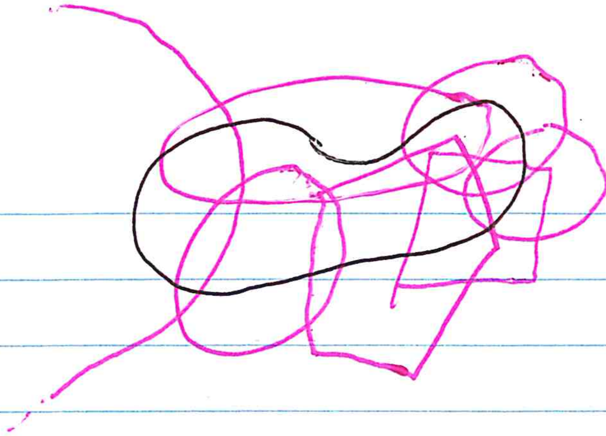
If all the member of a covering $\mathcal{U}$ of A are open, we say $\mathcal{U}$ is an <u>open covering</u> of A.

Ex The collection $\{(n, n+2) \mid n \in \mathbb{Z}\}$ is an open covering of $\mathbb{R}$.

The collection $\{\mathbb{R}\}$ is an open covering of $\mathbb{R}$.

The collection $\{(0, 1-\frac{1}{n}) \mid n \geq 2\}$ is an open covering of $(0,1)$.

The collection of open disks is an open covering of $\mathbb{R}^2$. The collection of open disks with rational radii and centers is an open subcovering.

Ex 

Def If every open covering of A has a finite subcovering, then A is <u>covering compact</u>.

We will show that for metric spaces covering compactness and sequencial compactness are the same. This is not true for more general topological spaces. In most books and journal article the word compact means covering compact.

Ex $\mathbb{R}$ is not compact because the open covering we give earlier has no finite subcover. Prove this!

Ex Show that $(0,1)$ is not covering compact.

Ex Show that $\{1\}$ is a compact subset of $\mathbb{R}$.

Thm Covering compactness is equivalent to seq. compactness.

We will prove one direction, take a detour, and then prove the converse.

CC ⇒ SC: Suppose $A \subset M$ is c.c. but not s.c. Then $\exists$ a seq. (p_n) in A with no convergent subseq. We use this to construct an open covering of A.

For each $a \in A$ $\exists$ an open ball with center a, B_a, whose radius is small enough that $B_a \cap \{p_n\}$ is a finite set (it could be empty). Then the collection $\{B_a \mid a \in A\}$ is an open covering of A.

Since A is c.c. $\exists$ a finite subcollection, $\{B_{a_1}, B_{a_2}, \dots, B_{a_n}\}$, that covers A. But the union of these balls has only finitely many entries from (p_n). Thus this does not cover A. Hence every seq in A does have a convergent subseq and so A is s.c. $\square$

Note: The seq (p_n) used above cannot have a value the occurs infinitely often since then it would have a convergent subseq.

Thm (Lebesgue number Lemma) Let A be a seq. comp. set in a metric space. Let $\mathcal{U}$ be an open covering of A. Then $\exists \lambda > 0$ s.t. $\forall a \in A$, $\exists u \in \mathcal{U}$ s.t. $\mathring{B}(a, \lambda) \subset \mathcal{U}$.

In other words, we know every pt in A is in some $\mathcal{U} \in \mathcal{U}$, but if A is seq comp, $\exists \lambda > 0$ s.t every open ball of radius $\le \lambda$ is in some $\mathcal{U} \in \mathcal{U}$. (The centers are in A, but the balls need not be contained in A.)

Def The number λ is called a Lebesgue number for the covering $\mathcal{U}$.

Note This is a purely metric idea and cannot be carried over to general topological spaces.

Pf Suppose not. Let $\mathcal{U}$ be an open covering of a seq. comp. set A s.t.

$$\forall \lambda > 0,\ \exists a \in A \text{ s.t. } \forall u \in \mathcal{U},\ \mathring{B}(a, \lambda) \not\subset \mathcal{U}. \quad (*)$$

We will construct a seq s.t. as the points cluster, as they must, an open ball about the cluster pt will be in some $\mathcal{U}$.

For $n = 1, 2, 3, \ldots$ let $\lambda = 1/n$ and let $a_n \in A$ be such a pt., a pt where $(*)$ holds. Consider the seq. (a_n). It must have a convergent subseq. (a_{n_k}). Let $p = \lim_{k\to\infty} a_{n_k}$.

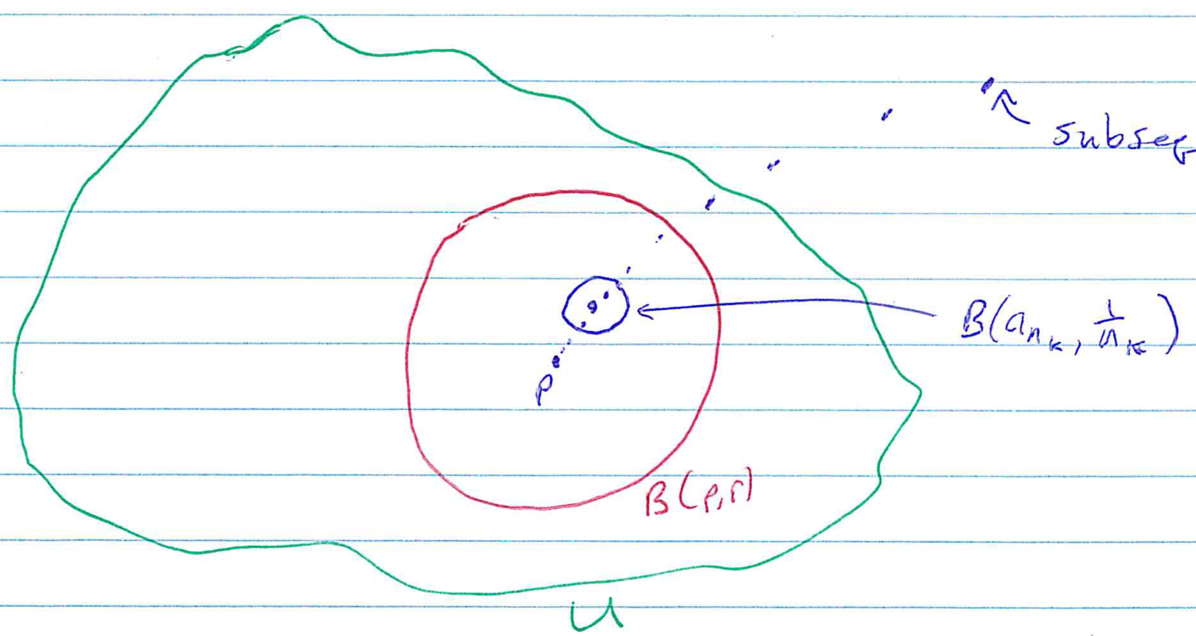
$\exists\, U \in \mathcal{U}$ with $p \in U$ and $\exists\, r > 0$ s.t. $\mathring{B}(a, r) \subset U$.

$\exists K$ s.t. $k > K \Rightarrow d(a_{n_k}, p) < r/2$ and $1/n_k < r/2$. Fix such a k.

You can use the triangle ineq to show that

$$\mathring{B}(a_{n_k}, 1/n_k) \subset \mathring{B}(p, r) \subset U.$$

But this is a contradiction. $\boxed{\checkmark}$



Now we prove that S.C. $\Rightarrow$ C.C.

Let $\mathcal{U}$ be an open covering of a compact set A. Let $\lambda > 0$ be a Leb. num. for $\mathcal{U}$. Thus $\forall a \in A$ $\exists u \in \mathcal{U}$ s.t. $\mathring{B}(a, \lambda) \subset U$.

Now we construct a sequence. Let $a_1 \in A$ and $U_1 \in \mathcal{U}$ be s.t. $\mathring{B}(a_1, \lambda) \subset U_1$. If $A \subset U_1$ then $\{U_1\}$ is a finite subcover and we are done.

If not, let $a_2 \in A - U_1$ and $U_2 \in \mathcal{U}$ s.t. $\mathring{B}(a_2, \lambda) \subset U_2$. If $A \subset U_1 \cup U_2$ then $\{U_1, U_2\}$ is a finite subcover.

If not, we contin this process, generating a seq (a_n) and members of $\mathcal{U}$, (U_n) s.t.

$$\mathring{B}(a_n, \lambda) \subset U_n.$$

If this process terminates at some n, that is if $\{U_1, \dots U_n\}$ is a subcover, we are done. If not (a_n) and (U_n) are infinite. We shall find a contradiction.

By S.C. $\exists$ a conv. subseq. (a_{n_k}) and $p = \lim_{k \to \infty} a_{n_k} \in A$.

$\exists K$ s.t. $k > K \Rightarrow d(a_{n_k}, p) < \lambda$. Therefore,

$$p \in \mathring{B}(a_{n_k}, \lambda) \subset U_{n_k} \quad \text{for } k > K.$$

But wait! Fix $k > K$. For $l > k$ $a_{n_l} \notin U_{n_k}$

but construction. This contradicts the def. of convergence. $\boxed{}$

Total Boundedness

<u>Def</u> A set A in a metric space M is **totally bounded** if $\forall\ \varepsilon > 0\ \exists$ a finite covering of A by open balls of radius ε.

Recall the Heine-Borel thm a subset of $\mathbb{R}^m$ is compact iff it is closed and bdd.*

This is false in general. In $\mathbb{Q}$ a closed bdd set need not be compact; $[\pi, \pi+1] \cap \mathbb{Q}$ is an example. This is because $\mathbb{Q}$ is not complete.

But completeness is not enough. $\mathbb{N}$, with the metric $d(n,m) = 1$ $m \neq n$, 0 $m = n$, is closed, bdd and complete. But it is not compact. The collection $\{\{n\}\}_{n=1}^{\infty}$ is an open covering with no finite subcover.

* H-B gives only one direction. The other was given in the same section. See 79-81.

It turns out the total boundedness is the missing ingredient.

Thm Generalized Heine-Borel Thm.
A subset of a complete metric space is compact if and only if it is closed and totally bdd.

Corollary A metric space is compact iff it is complete and totally bdd.

See the textbook, pgs 103-104, for the proofs.

Optional Reading/Listening

I mentioned before that seq. and cov. compactness are not equivalent on all topological spaces. Why is it that covering compactness is considered the "prefered" definition of compactness? It has to do, in part, with the properties of infinite products of spaces. It turns out that the "natural" topology on such products preserves covering compactness but not necessarly seq. compactness. Look up the Tychonoff thm in any graduate level topology textbook.

There are several other notions of compactness that have been studied. A top sp is Lindelöf if every open cover has a ~~finite~~ countable subcover. A top sp is countably compact if every countable open cover has a finite subcover. Other notitions include σ-compact, hemicompact, pseudocompact, realcompact and paracompact.