# Chapter 3: Functions of one real variable

**Section 1** I assume you know basic facts about derivatives and how to prove them. You should review your old calculus textbook and think about how you would teach calc; how would you balance theory and practice, rigor and intuition?

**Note** Most calc. book prove $(x^n)' = n x^{n-1}$ by using the binomial thm. This is bad. Instead do the prod. rule first, then use induction.

**Note** The text says the MVT is important for error estimates. This is very true. See my webpage for a handout on the proof of the error estimation formula for the trapizoidal rule. I use it as optional reading in my calc II classes. For you it is required reading.

**Note** Some proofs give us intuitive insight into why the thm is true; some do not. For basic calc. results you should not merely know the proofs, but you should be able to give calc students an intuitive feel for them. Here are two examples. Come up with some of your own.

Ex (The Chain Rule) If Sue can run twice as fast as Bill, and Bill can run three times as fast as Jane, then we know that Sue can run six times as fast as Jane.

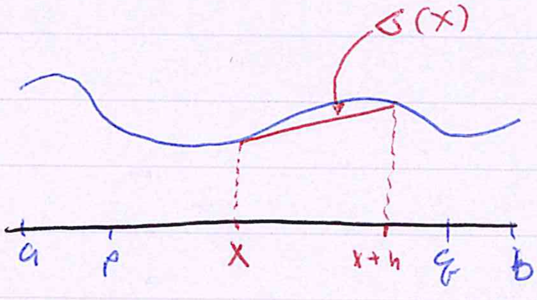
Ex (The MVT or what it should be called, the Average Speed Thm.) If Grandma's house is 100 miles away and you get there in one hour, she knows you were speeding.

Here is something you might not know

**Thm 8 (pg 154)** If $f$ is differentiable on $(a,b)$ then its derivative function $f'(x)$, although not necessarily continuous, has the intermediate value property.

**Pf** Suppose $a < p < q < b$ and $\alpha = f'(p) < \gamma < f'(q) = \beta$. (The other case, $q < p$, is similar.)

Choose $h \in (0, p-q)$ and consider the line segments, denoted $\sigma(x)$, from $(x, f(x))$ to $(x+h, f(x+h))$ as $x$ ranges from $p$ to $q-h$. Note: $\sigma : [p, q-h] \longmapsto$ {the set of line segments of $\mathbb{R}^2$.




Let $s(x) =$ slope of $\sigma(x)$. Thus, $s(x) = \dfrac{f(x+h) - f(x)}{h}$

and so is a cont. func. from $[p, q-h]$ to $\mathbb{R}$.

Let $0 < \varepsilon < \min\{|f'(p) - \gamma|, |f'(q) - \gamma|\}$.

$\exists\, \delta > 0$ s.t. $0 < h < \delta \Rightarrow |s(p) - f'(p)| < \varepsilon$ and

$|s(q-h) - f'(q)| < \varepsilon$, since $\lim_{h \to 0} s(x) = f'(x)$.

Fix such an $h$. Then

$$s(p) < \gamma < s(q-h).$$

Since $s$ is cont. $\exists\, x_0 \in (p, q-h)$ s.t. $s(x_0) = \gamma$.

Apply the MVT to $f$ over $(x_0, x_0+h)$ to

get $\exists\, r \in (x_0, x_0+h)$ s.t.

$$f'(r) = \frac{f(x_0+h) - f(x_0)}{h} = s(x_0) = \gamma.$$

$\blacksquare$

Some notation for function sets/spaces.

$C^0 = C^0(X, Y)$ = conti. functions from $X$ to $Y$.
Usually $X$ = some real interval and $Y = \mathbb{R}$.

$C^1((a,b), \mathbb{R}) = \{ f:(a,b) \to \mathbb{R} \mid f' \text{ exists and is cont.} \}$

$C^2$ $\qquad\qquad\qquad\qquad f''$

$C^3$ $\qquad\qquad\qquad\qquad f'''$

etc

$C^\infty \quad = \{ \qquad\qquad \forall n, f^{(n)}, \qquad \}$

<u>Def</u> A function $f:(a,b) \to \mathbb{R}$ is <u>analytic</u> if $\forall x_0 \in (a,b)$ $\exists \delta > 0$ s.t. if $|\Delta x| < \delta$ then

$$f(x_0 + \Delta x) = \sum_{n=0}^{\infty} a_n (\Delta x)^n$$

$C^\omega$ = set of analytic functions.

Taylor's Thm gives a formula for the $a_n$'s.

$C^0 \supset C^1 \supset C^2 \supset \cdots \supset C^\infty \supset C^\omega$, and each inclusion is proper.

In Exercise 17 (pg 200) you will show $\exists$ $C^\infty$ functions that are not analytic.

## $\boxed{\text{Taylor Polynomials and Approximation}}$

**Def** Let $f:(a,b)\to\mathbb{R}$ be in $C^n$ for some $n\geq 1$, $n=\infty$ or $\omega$ is allowed. Let $x_0\in(a,b)$. Then the $n^{th}$ order Taylor poly of centered at $x_0$ is

$$P_n(x)=\frac{f(x_0)}{0!}+\frac{f'(x_0)}{1!}(x-x_0)^1+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

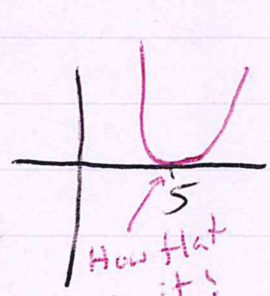
$$=\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k.$$

**Def** A function $f:(a,b)\to\mathbb{R}$ is $n^{th}$ order flat at $x=x_0$ if

$$\lim_{h\to 0}\frac{f(x_0+h)}{h^n}=0.$$

**Ex** Let $f(x)=(x-5)^4$. Then $f(x)$ is $3^{rd}$ order flat at $x=5$, since,

$$\lim_{h\to 0}\frac{(5+h-5)^4}{h^3}=\lim_{h\to 0}h=0$$



**Thm** $\boxed{\text{Taylor's Approximation Thm}}$

Suppose $f:(a,b)\to\mathbb{R}$ is $n^{th}$ order diff. at $x_0$. Then the following hold.

ⓐ Let $R(x) = f(x) - P_n(x)$. Then $R(x)$ is $n^{th}$ order flat at $x_0$.

ⓑ $P_n(x)$ is the only poly of degree $\le n$ for which ⓐ holds.

ⓒ If $f$ is $n+1$ diff. on $(a,b)$, $\forall x\in(a,b)$ then $\exists\,\theta$ in between $x_0$ and $x$ s.t.

$$R(x) = \frac{f^{(n+1)}(\theta)}{(n+1)!}(x-x_0)^{n+1}.$$

$\boxed{\text{pf}}$ ⓐ All derivatives of $P_n(x)$ exist, so $R(x)$ has at least $n$ derivatives. It is easy to check $R^{(k)}(x_0)=0$ for $k=0,\dots,n$. Let $x=x_0$,

$$R(x_0) = f(x_0) - \left[f(x_0) + f'(x_0)(x_0-x_0) + \frac{f''(x_0)}{2}(x_0-x_0)^2 + \dots + \frac{f^{n}(x)}{n!}(x_0-x_0)^n\right]$$

$$= f(x_0) - f(x_0) = 0.$$

$$R'(x_0) = f'(x_0) - \left[f'(x_0) + f''(x_0-x_0) + \frac{f'''(x_0)}{2}(x_0-x_0)^2 + \dots \frac{f^{n}(x_0)}{n!}(x_0-x_0)^{n-1}\right]$$

$$= f'(x_0) - f'(x) = 0$$

$$R''(x_0) = f''(x_0) - \left[f''(x_0) + f'''(x_0)(x_0 - x_0) + \cdots + \frac{f^{n}(x_0)}{(n-2)!}(x_0 - x_0)^{n-2}\right] = 0$$

$\vdots$

$$R^{(n)}(x_0) = f^{(n)}(x_0) - f^{n}(x_0) = 0.$$

Suppose $x_0 < X$. (The other case is similar.)
By the MVT $\exists\, \theta_1 \in (x_0, X)$ s.t.

$R'(\theta_1) = \dfrac{R(X) - \cancel{R(x_0)}^{\,0}}{X - x_0}$. Thus, $R'(\theta_1)(X - x_0) = R(X)$.

By the MVT $\exists\, \theta_2 \in (x_0, \theta_1)$ s.t.

$R''(\theta_2) = \dfrac{R'(\theta_1) - \cancel{R'(x_0)}^{\,0}}{\theta_1 - x_0}$. Thus, $R''(\theta_2)(\theta_1 - x_0) = R'(\theta_1)$.

Hence, $R''(\theta_2)(\theta_1 - x_0)(X - x_0) = R(X)$.

By the MVT $\exists\, \theta_3 \in (x_0, \theta_2)$ s.t.

$R'''(\theta_3) = \dfrac{R''(\theta_2) - \cancel{R''(x_0)}^{\,0}}{\theta_2 - x_0}$. Thus, $R'''(\theta_3)(\theta_2 - x_0) = R''(\theta_2)$.

Hence, $R'''(\theta_3)(\theta_2 - x_0)(\theta_1 - x_0)(X - x_0) = R(X)$.

$\vdots$

$$R^{(n-1)}(\theta_{n-1})(\theta_{n-2} - x_0)(\theta_{n-3} - x_0)\cdots(\theta_1 - x_0)(X - x_0) = R(X)$$

for some $\theta_i$'s with $x_0 < \theta_{n-1} < \theta_{n-2} < \cdots < \theta_1 < X$.

Now;

$$\left| \frac{R(x)}{(x-x_0)^n} \right| = \left| \frac{R^{(n-1)}(\theta_{n-1})(\theta_{n-2}-x_0)\cdots(x-x_0)}{(x-x_0)^n} \right|$$

$$\leq \left| \frac{R^{(n-1)}(\theta_{n-1})}{x-x_0} \right|$$

$$\leq \left| \frac{R^{(n-1)}(\theta_{n-1})}{\theta_{n-1}-x_0} \right| \longrightarrow \left| R^{(n)}(x_0) \right| = 0$$

limit as $x \to x_0$ from the right,
forces $\theta_{n-1} \to x_0$ " " " .

As said before the case $x_0 > x$ is similar. The last limit will be from the left. $\boxed{\text{(a) is proved}}$

(b) This is easy. See textbook.

(c) Suppose $x > x_0$. (The case for $x < x_0$ is similar.)
For $t \in [x_0, x]$ define

$$g(t) = f(t) - P(t) - \frac{R(x)(t-x_0)^{n+1}}{(x-x_0)^{n+1}}.$$

We will show that at the end points $g(x_0) = g(x) = 0$ and that $g^{(k)}(x_0) = 0$ for $k = 1, 2, 3, \dots, n$. We will also compute $g^{(n+1)}(t)$.

$$g(x_0) = \underbrace{f(x_0) - P(x_0)}_{\text{Those cancel.}} - \frac{R(x)(\overbrace{x_0 - x_0}^{=0})^{n+1}}{(x-x_0)^{n+1}} = 0$$

$$g(x) = f(x) - P(x) - \frac{R(x)(x-x_0)^{n+1}}{(x-x_0)^{n+1}} =$$

$$f(x) - P(x) - R(x) = 0, \text{ by def of } R.$$

Now for the derivatives. We apply $\frac{d}{dt}$, but remember $x$ is a constant for now.

$$g'(t) = f'(t) - P'(t) - \frac{(n+1)R(x)(t-x_0)^n}{(x-x_0)^{n+1}}$$

$$g'(x_0) = \underbrace{f'(x_0) - P'(x_0)}_{\text{cancel}} - \frac{(n+1)R(x)(\overbrace{x_0 - x_0}^{0})^n}{(x-x_0)^{n+1}} = 0$$

$$g''(t) = f''(t) - p''(t) - \frac{(n+1)n\,R(x)\,(t - x_0)^{n-1}}{(x - x_0)^{n+1}}$$

$$g''(x_0) = \underbrace{f''(x_0) - p''(x_0)}_{\text{cancel}} - \frac{(n+1)n\,R(x)\,(\overbrace{x_0 - x_0}^{0})^{n-1}}{(x - x_0)^{n+1}} = 0.$$

$$\vdots$$

$$g^{(n)}(t) = f^{(n)}(t) - p^{(n)}(t) - \frac{(n+1)!\,R(x)\,(t - x_0)^{1}}{(x - x_0)^{n+1}}$$

$$g^{(n)}(x_0) = \underbrace{f^{(n)}(x_0) - p^{(n)}(x_0)}_{\text{cancel}} - \frac{(n+1)!\,R(x)\,(\overbrace{x_0 - x_0}^{0})}{(x - x_0)^{n+1}} = 0$$

And finally,

$$g^{(n+1)}(t) = f^{(n+1)}(t) - 0 - \frac{(n+1)!\,R(x)}{(x - x_0)^{n+1}}.$$

Now the $\boxed{\text{MVT}}$ comes in and saves the day!

MVT $\Rightarrow$ $\exists_{\theta_1 \in (x_0, x)}$ s.t. $g'(\theta_1) = \dfrac{g(x) - g(x_0)}{x - x_0} = 0.$

MVT $\Rightarrow$ $\exists_{\theta_2 \in (x_0, \theta_1)}$ s.t. $g''(\theta_2) = \dfrac{g'(\theta_1) - g'(x_0)}{\theta_1 - x_0} = 0$

MVT $\Rightarrow$ $\exists_{\theta_3 \in (x_0, \theta_2)}$ s.t. $g'''(\theta_3) = \dfrac{g''(\theta_2) - g''(x_0)}{\theta_2 - x_0} = 0.$

$\vdots$

MVT $\Rightarrow$ $\exists_{\theta_{n+1} \in (x_0, \theta_n)}$ s.t. $g^{(n+1)}(\theta_{n+1}) = \dfrac{g^{(n)}(\theta_n) - g^{(n)}(x_0)}{\theta_n - x_0} = 0.$

Recall we had a formula for $g^{(n+1)}(t)$. It gives

$$f^{(n+1)}(\theta_{n+1}) - \frac{(n+1)!\, R(x)}{(x - x_0)^{n+1}} = 0.$$

Now we simply solve for $R(x)$.

$$R(x) = \frac{f^{(n+1)}(\theta_{n+1})(x - x_0)^{n+1}}{(n+1)!}.$$

This is the formula in part c.

$\boxed{\star}$