Section 2: Riemann Integration

Def A partition of $[a,b]$ is a finite set $P = \{x_0, x_1, \dots, x_n\}$ s.t. $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. A set of test points or sample points for a partition P is set $T = \{t_1, t_2, \dots, t_n\}$ s.t. $x_{i-1} \le t_i \le x_i$, $i = 1, \dots, n$. Let $\Delta x_i = x_i - x_{i-1}$. The mesh of a partition is $\max\limits_{1 \le i \le n}\{\Delta x_i\}$.

Def Let $f : [a,b] \to \mathbb{R}$. Let P be a partition of $[a,b]$ and let T be a set of test points for P. Then the Riemann Sum of f w.r.t. P and T is

$$R(f, P, T) = \sum_{i=1}^{n} f(t_i)\, \Delta x_i .$$

Def Let $f : [a,b] \to \mathbb{R}$. If $\exists\, I \in \mathbb{R}$ s.t. $\forall \varepsilon > 0$, $\exists\, \delta > 0$ s.t. if P is any partition of $[a,b]$ with mesh $< \delta$, and for any T we have

$$|R(f, P, T) - I| < \varepsilon,$$

then we say I is the Riemann integral of f over $[a,b]$ and write

$$I = \int_a^b f(x)\, dx.$$

When the integral exists we say f is Riemann integrable over $[a,b]$.

Q: Let $\mathcal{R}$ be the set of Riemann integrable functions over $[a,b]$. What can we say about $\mathcal{R}$? Which functions are in $\mathcal{R}$? What structural properties does $\mathcal{R}$ have?

Thm: Every Riemann integrable function over $[a,b]$ is bounded.

Outline of Pf: Suppose f is unbdd but that I exists. Let $\varepsilon = 1$. Let $\delta > 0$. Choose any partition with mesh $< \delta$. $\exists\ [x_k, x_{k+1}]$ on which f is unbdd. Then show that for some $t \in [x_k, x_{k+1}]$, when used as a sample point

$$|R(f,P,T) - I| > \varepsilon = 1.$$

~~Def~~ Ex Let $f(x) = \frac{1}{x}$ for $x = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots$, and zero elsewhere. Then $\int_0^1 f(x)\,dx$ does not exist. Is that the best we can do?

Def: Let $S \subset \mathbb{R}$. The characteristic function of S is

$$\chi_S = \begin{cases} 1 & x \in S \\ 0 & x \notin S. \end{cases}$$

Ex $\chi_{\mathbb{Q}} \notin \mathcal{R}$.* Pf: Let $\varepsilon = \frac{1}{2}$. For any partition P we can choose rational pts for T and get $R(\chi_{\mathbb{Q}}, P, T) = 1$ or irrational pts and gets $R(\chi_{\mathbb{Q}}, P, T) = 0$.

* Over any interval $[a,b]$.

<u>Fact</u> $\mathcal{R}$ is a vector space under addition of functions and scalar mult. by real numbers. The proof, see textbook, just involves showing

$$\int_a^b f+g\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

and $$\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx.$$

This can also be interpreted to mean that

$$f \longmapsto \int_a^b f(x)\,dx$$

is a linear transformation from the vector space $\mathcal{R}$ to the v. sp. $\mathbb{R}$.

See textbooks for other basic properties of integration.

Darboux Integrability

This will prove to be a useful tool for determining Riemann integrability.

Def: Let $f:[a,b] \to [-M, M]$, so f is bdd. Let $P = \{x_0, x_1, \ldots x_n\}$ be a partition of $[a,b]$. In each subinterval let

$$m_i := \inf\{f(t) : t \in [x_{i-1}, x_i]\} \text{ and } M_i := \sup\{f(t) : t \in [x_{i-1}, x_i]\}.$$

Note: f need not have a min or max in $[x_{i-1}, x_i]$ since it need not be cont. **Ex:** $f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ x & x \notin \mathbb{Q} \end{cases}$. Then max f on $[1,2]$ does not exist, but $\sup = 2$.

Now, define $\boxed{L(f,P) = \sum_{i=1}^{n} m_i \Delta x_i}$ and $\boxed{U(f,P) = \sum_{i=1}^{n} M_i \Delta x_i}$.

Then for any T, $L(f,P) \le R(f,P,T) \le U(f,P)$.

Let $\underline{I} = \sup\{L(f,P) : P \in \mathcal{P}\}$ and

$\overline{I} = \inf\{U(f,P) : P \in \mathcal{P}\}$,

where $\mathcal{P}$ = all partitions of $[a,b]$. If $\underline{I} = \overline{I}$, then we say f is **Darboux integrable** and $\underline{I} = \overline{I}$ is the **Darboux integral** of f over $[a,b]$.

Thm (20) Riemann integrability and Darboux integrability are the same. If $\underline{I} = \overline{I}$, then $\underline{I} = I = \overline{I}$.

Before proving this we give two applications. The first is easy.

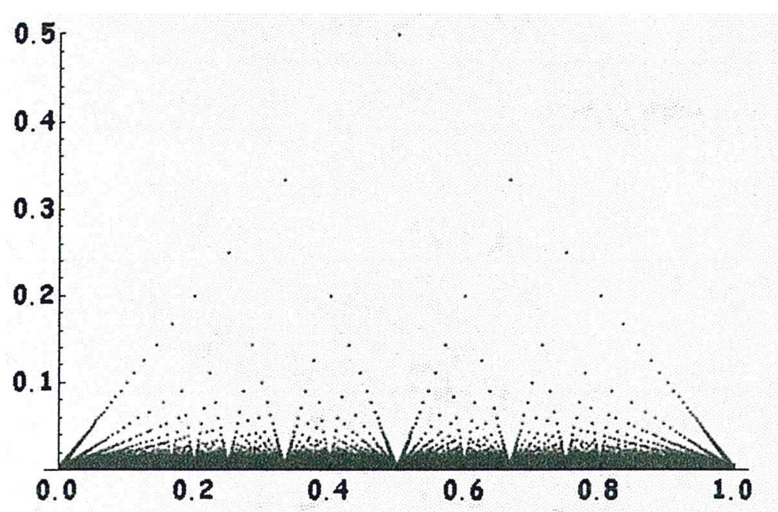
Ex It is now immediate that $L(\chi_{\mathbb{Q}}, P) = 0$ and $U(\chi_{\mathbb{Q}}, P) = 1$ for all partitions. Thus $\chi_{\mathbb{Q}} \notin \mathcal{R}$.

Ex We define the rational ruler function, $f : [0,1] \to \mathbb{R}$, as follows. Let

$$f(x) = \begin{cases} 0 & x \text{ irrational} \\ \frac{1}{q} & \text{for } x = \frac{p}{q}, \text{ expressed in reduced form.} \end{cases}$$

We take it that $f(0) = f(1) = 1$.

Thus, $f(\frac{1}{\pi}) = 0$, $f(\frac{3}{7}) = \frac{1}{7}$, $f(\frac{3}{9}) = f(\frac{1}{3}) = \frac{1}{3}$, and so on.



From Math Is Fun Forum.

Thm The rational ruler function is Darboux integrable and $\underline{I} = \overline{I} = 0$.

Pf For any partition P it is clear that $L(f,P) = 0$, since each $m_i = 0$ (since each subinterval contains an irrational number). Let $\varepsilon > 0$. We will find a partition P s.t. $U(f,P) < \varepsilon$. Then $\underline{I} = \overline{I} = 0$.

We will form a seq of partitions (P_k). First, let $A_k = \{ \frac{p}{q} \in [0,1]_{\mathbb{Q}} \mid q = 1, 2, 3, \dots K \}$ Let Δ_k be the smallest gap between members of A_k. For example,

$$A_3 = \{0, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, 1\} \text{ and } \Delta_k = 1/6.$$

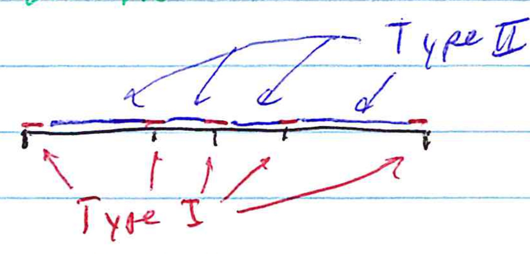
Suppose the members of A_k are enumerated in increasing order as $a_0, a_1, a_2, a_3, \dots a_n$. Let $\delta_k = \frac{\Delta_k}{2(n+1)^2}$. Note that as $k \to \infty$, $n \to \infty$. Define

$$P_k = \{a_0, a_0 + \delta_k, a_1 - \delta_k, a_1 + \delta_k, a_2 - \delta_k, a_2 + \delta_k, \dots, a_n - \delta_k, a_n\}.$$

These determine two families of subintervals of $[0,1]$.

Type I: Those that contain an a_i and
Type II: those that do not.

Ex $K = 3$
$\delta_k = \frac{1}{12 \cdot 6^2}$



On type II intervals, $\sup f \le \frac{1}{k}$. Their total length is obviously < 1. Thus their contribution to $U(f, P_k)$ is $< \frac{1}{k}$.

On type I intervals all we can say about $\sup f$ is that it is ≤ 1. But they are very thin. Their total length is

$$(2\delta_k)n = \frac{2\Delta_k n}{2(n+1)^2} \le \frac{n}{(n+1)^2} < \frac{1}{n+1}.$$

Thus the contribution of type I intervals to $U(f, P_k)$ is $< \frac{1}{n+1}$. Therefore,

$$U(f, P_k) < \frac{1}{k} + \frac{1}{n+1} < \frac{2}{k}.$$

We can choose k so that $\frac{2}{k} < \varepsilon$. $\square$

Note The textbook gives a different and much shorter proof. See if you can follow it. The proof here illustrates an important technique: Surround the "bad pts" with very thin partition members and watch them fade away.

The book also pts out that the rational rule function is discontinuous on $\mathbb{Q}$, but is continuous off of $\mathbb{Q}$. Could it be differentiable there as well?

To prove Thm (20) we need two definitions and some Lemmas.

Def If P is a partition of $[a,b]$ and P' is another partition of $[a,b]$, we say P' <u>refines</u> P if $P \subset P'$. Clearly mesh $P' \leq$ mesh P.

Def Given two partitions, P_1 and P_2, of $[a,b]$, their <u>common refinement</u> is $P_1 \cup P_2$.

Lemma If P' refines P then $L(f,P') \geq L(f,P)$ and $U(f,P') \leq U(f,P)$.

Pf Obvious.

Lemma For any two partitions, P_1 and P_2, of $[a,b]$ we have

$$L(f,P_1) \leq U(f,P_2).$$

Pf $L(f,P_1) \leq L(f,P_1 \cup P_2) \leq U(f,P_1 \cup P_2) \leq U(f,P_2)$.

Corollary (☆) $\underline{I} = \overline{I} \iff \forall \varepsilon > 0 \ \exists$ partition P s.t. $U(f,P) - L(f,P) < \varepsilon$.

~~Pf~~

Pf $\underline{I} = \bar{I} \iff \forall \varepsilon > 0 \ \exists$ partitions P, P' s.t.

$$|U(f,P) - L(f,P')| < \varepsilon.$$

Clearly, we can drop the absolute value signs. If we let $P^* = P \cup P'$, then combining the two lemmas

$$L(f,P') \le L(f,P^*) \le U(f,P^*) \le U(f,P).$$

The result follows. $\blacksquare$

Proof that Darboux $\Longleftrightarrow$ Riemann.

Let $f \in \mathcal{R}$ and $I = \int_a^b f(x)\,dx$. We know f is bounded. Let $\varepsilon > 0$. $\exists\ \delta > 0$ s.t. $\forall P, T$ with mesh $< \delta$

$$|R - I| < \frac{\varepsilon}{4}.$$

Fix a partition P with mesh $< \delta$. In each $[x_{i-1}, x_i]$ $\exists\ t_i$ s.t. $f(t_i) - m_i < \eta$, where $\eta > 0$ is arbitrary for now. Let $T = \{t_1, \dots, t_n\}$. Then

$$R(f,P,T) - L(f,P) = \sum f(t_i)\Delta x_i - \sum m_i \Delta x_i$$

$$= \sum (f(t_i) - m_i)\Delta x_i < \sum \eta \Delta x_i = \eta(b-a).$$

Choose $\eta < \frac{\varepsilon}{4(b-a)}$. Now

$$R(f,P,T) - L(f,P) < \frac{\varepsilon}{4}.$$

Similarly, $\exists\ T' = \{t_1', \dots, t_n'\}$ s.t.

$$U(f,P) - R(f,P,T') < \frac{\varepsilon}{4}.$$

Since mesh $P < \delta$ we have

$$U - L = (U - R') + (R' - I) + (I - R) + (R - L) < \varepsilon.$$

Thus, $\forall \varepsilon > 0$, $\exists P$ s.t.

$$L < \underline{I} < \overline{I} < U.$$

Hence, $\overline{I} - \underline{I} < \varepsilon$, $\forall \varepsilon > 0$. Thus, $\overline{I} = \underline{I}$.
Since $\underline{I} \le I \le \overline{I}$, they are equal.

Assume $f : [a,b] \to [-M, M]$ is Darboux int. over $[a,b]$. Let $I = \underline{I} = \overline{I}$. To show $f \in \mathcal{R}$ we will work with three partitions of $[a,b]$:

$$P_1 = \{y_0, y_1, y_2, \dots, y_{n_1}\},$$

$$P = \{x_0, x_1, x_2, \dots, x_n\},$$

$$P^* = P_1 \cup P = \{x_0^*, x_1^*, x_2^*, \dots, x_{n^*}^*\}.$$

Let $\varepsilon > 0$. By ($\star$) we can assume P_1 is s.t.

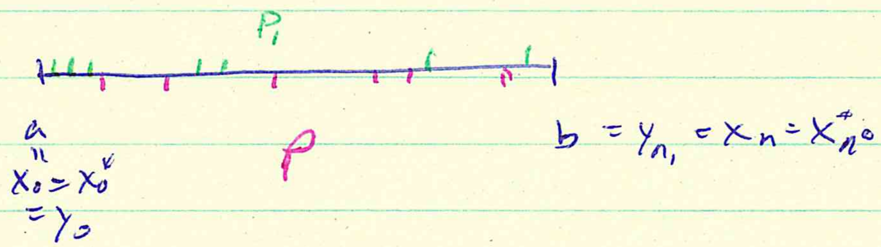
$$U_1 - L_1 < \varepsilon/2.$$

Let $\delta = \frac{\varepsilon}{8Mn_1}$. Let P be a partition with mesh $< \delta$. Let $P^* = P \cup P_1$. By Lemma 1

$$L_1 \le L^* \le U^* \le U_1.$$

Thus $U^* - L^* < \varepsilon/2$. Now we compare the terms of

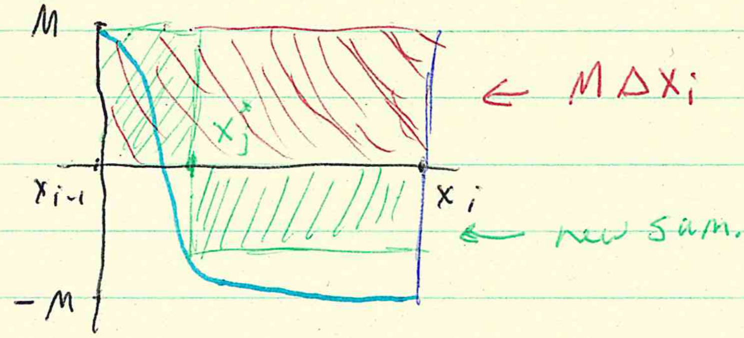
$$U = \sum M_i \Delta x_i \quad \text{and} \quad U^* = \sum M_i^* \Delta x_i^*.$$



If there is no x_j^* in (x_{i-1}, x_i) then the term $M_i \Delta x_i$ appears in both sums, but with possibly different indices. Since the endpoints of P^* and P are the same there are at most $n_1 - 1$ members of $\{(x_{i-1}, x_i)\}$ that do not contain an x_j^*. For each of those

$$|M_i| \Delta x_i < M\delta.$$

When we partition $[x_{i-1}, x_i]$ with an x_j^* in its interior what is the maximum impact on the sum? It is $2M\delta$. See Figure



Therefore,

$$U - U^* \le (n_1 - 1)\, 2M\delta = (n_1 - 1)\, 2M \frac{\varepsilon}{8Mn_1}$$

$$= \frac{n_1 - 1}{n_1} \frac{\varepsilon}{4} < \frac{\varepsilon}{4}.$$

Similarly $L^* - L < \frac{\varepsilon}{4}$. Thus,

$$U - L = (U - U^*) + (U^* - L^*) + (L^* - L) < \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon.$$

Since, $L \le \underline{I} = I = \overline{I} \le U$ and $L \le R(f,P,T) \le U$ we have $|R - I| < \varepsilon$. Thus $f \in \mathcal{R}$ and

$$\int_a^b f(t)\,dt = I.$$

Thm (23, page 175) The Riemann-Lebesgue Thm

A function $f: [a,b] \to \mathbb{R}$ is Riemann integrable iff it is bdd and the set of discontinuities is a zero set.

Def Recall a set $Z \subset \mathbb{R}$ is a zero set (or has "Lebesgue measure*" zero) if for $\forall \varepsilon > 0$ $\exists$ a countable covering of open intervals $\{(a_i, b_i)\}_{i=1}^{\infty \text{ or } n}$ s.t.

$$\sum b_i - a_i < \varepsilon.$$

Ex Finite sets, countable sets, the middle thirds Cantor set.

Any subset of a zero set is a zero set.

Any countable union of zero sets is a zero set.

See textbook for proof.

Befor proving the RLT we need some more definitions leading to means of looking at the size of a discontinuity.

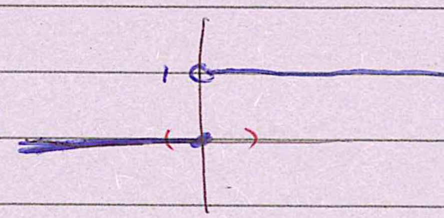
*See page 383 and 386

Def Let $f : [a,b] \to \mathbb{R}$ and let $x \in (a,b)$.

$$\limsup_{t \to x} f(t) = \lim_{\varepsilon \to 0} \sup\{ f(t) \mid t \in (x-|\varepsilon|, x+|\varepsilon|)\}.$$

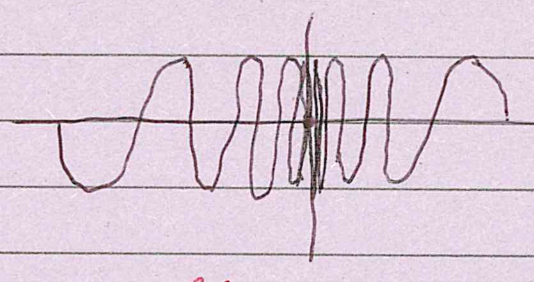
$$\liminf_{t \to x} f(t) = \lim_{\varepsilon \to 0} \inf\{ f(t) \mid t \in (x-|\varepsilon|, x+|\varepsilon|)\}.$$

Ex Let $f(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases}$.



Then $\limsup\limits_{t \to 0} f(t) = 1$ and $\liminf\limits_{t \to 0} f(t) = 0$.

Ex Let $f(x) = \sin\left(\frac{1}{x}\right)$ for $x \neq 0$ and $= 0$ at $x = 0$.



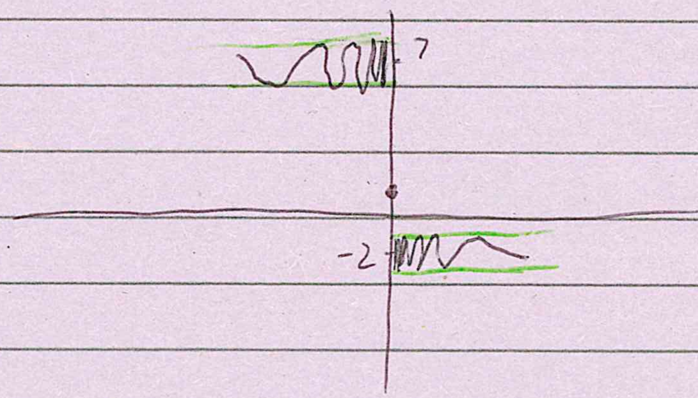
Then $\limsup\limits_{t \to 0} f(t) = 1$ and $\liminf\limits_{t \to 0} f(t) = -1$.

Ex Let $f(x) = 0$ for $x \neq 0$ and $f(0) = 13.7$. Then $\limsup\limits_{t \to 0} f(t) = 13.7$ and $\liminf\limits_{t \to 0} f(t) = 0$.

Def The oscillation of $f:(a,b)\to\mathbb{R}$ at x is

$$\operatorname{osc}_x(f) = \limsup_{t\to x} f(t) - \liminf_{t\to x} f(t).$$

Ex Let $f(x) = 7+\sin(\frac{1}{x})$ for $x<0$, $-2+\sin(\frac{1}{x})$ for $x>0$ and $f(0)=1$.



Then $\operatorname{osc}_0(f) = 8-(-3) = 11.$

Fact f is cont. at x $\iff$ $\operatorname{osc}_x(f)=0$.

Now we are ready to prove the RLT. The proof with use the Lebegue Lemma, so you should review it.

Proof of RLT

Let D be the set of pts in $[a,b]$ where f is not cont. For each $K \in \mathbb{N}$ let

$$D_K = \{x \in [a,b] \mid osc_x(f) \geq \frac{1}{K}\}.$$

Then $D = \bigcup_{K=1}^{\infty} D_K$, a countable union. Thus,

D is a zero set iff each D_K is a zero set.

($\Rightarrow$) Suppose $f \in \mathcal{R}$. Then it is bdd, so we may assume $f([a,b]) \subset [-M, M]$ for some $M > 0$.

Let $\varepsilon > 0$ and $K \in \mathbb{N}$. Then $\exists$ a partition P of $[a,b]$ s.t.

$$U - L = \sum (M_i - m_i)\Delta x_i < \frac{\varepsilon}{K}.$$

If $[x_{i-1}, x_i]$ contains a pt of D_K in its interior, then $M_i - m_i \geq \frac{1}{K}$. Call these intervals Type I intervals of P. We have

$$\frac{1}{K}\sum_{\text{Type I}} \Delta x_i \leq \sum_{\text{Type I}} (M_i - m_i)\Delta x_i \leq \sum_{\substack{\text{all intervals} \\ \text{for } P}} (M_i - m_i)\Delta x_i < \frac{\varepsilon}{K}.$$

Thus, $\sum_{\text{Type I}} \Delta x_i < \varepsilon$, that is the total length of these Type I intervals is small than ε.

Then D_k, except for possibly a finite number of pts, is covered by $\{(x_{i-1}, x_i)\}_{\text{Type I}}$ with total length $< \varepsilon$. Now ε was independent of k, thus D_k is a zero set. Therefore D is a zero set.

($\Leftarrow$) Now for the converse. Assume D is a zero set and that for some $M > 0$, $f([a,b]) \subset [-M, M]$.

Let $\varepsilon > 0$. Choose $k \in \mathbb{N}$ s.t. $\frac{1}{k} < \frac{\varepsilon}{2(b-a)}$.

Since $D_k \subset D$ it is a zero set. Therefore $\exists$ a countable open covering of D_k by open intervals, $\{J_i\}$, with total length $\leq \varepsilon/4M$.

$\forall\, x \in [a,b] - D_k$ $\exists$ an open interval I_x containing x s.t.

$$\sup\{f(t) \mid t \in I_x\} - \inf\{f(t) \mid t \in I_x\} < \frac{1}{k}.$$

Let $\mathcal{V} = \{J_i\}_{i=1}^{\infty} \cup \{I_x \mid x \in [a,b] - D_k\}$. ← might be finite

It is an open cover of $[a,b]$. Let $\lambda > 0$ be its Lebesgue number.

Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be any partition of $[a,b]$ with mesh $< \lambda$. We claim $U(f,P) - L(f,P) < \varepsilon$.

Each $[x_{i-1}, x_i]$ is inside some member of $\mathcal{V}$. Let

$$\mathbf{J} = \{i \in \{1, \ldots n\} \mid [x_{i-1}, x_i] \subset J_p \text{ for some } p\}.$$

$\mathbf{J}$ is a list of subintervals for P where $osc_x(f)$ is $\geq \frac{1}{k}$. (A list where big jumps occur.)

Let $m = \max \mathbf{J}$; it could be n. Then

$$U - L = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i = \sum_{i \in \mathbf{J}} (M_i - m_i)\Delta x_i + \sum_{i \notin \mathbf{J}} (M_i - m_i)\Delta x_i$$

$$\leq \sum_{i \in \mathbf{J}} 2M\Delta x_i + \sum_{i \notin \mathbf{J}} \frac{1}{k}\Delta x_i \leq 2M \sum_{i=1}^{m} \Delta x_i + \frac{b-a}{k}$$

$$\leq 2M\left(\frac{\varepsilon}{4M}\right) + \frac{\varepsilon}{2(b-a)}(b-a) = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $f \in \mathcal{R}$.

Implications

24 $C^0 \subset \mathcal{R}$. Piecewise cont. bdd function are in $\mathcal{R}$.

25 Let $S \subset [a,b]$. Then $\chi_S \in \mathcal{R}$ iff ∂S has measure zero (is a zero set).

Ex Let C be the Cantor middle thirds set and F be the middle forths set, both in $[0,1]$. Then $\chi_C \in \mathcal{R}$ but $\chi_F \notin \mathcal{R}$. See exercise 33.

26 Every monotone function on $[a,b]$ is in $\mathcal{R}$.

pf Let $f:[a,b] \to \mathbb{R}$ be monotone increasing. Then $\forall\, x \in [a,b]$ we have $f(a) \le f(x) \le f(b)$, so f is bdd. Suppose $a < x_1 < x_2 < x_3 < \dots < b$ and that $osc_{x_n}(f) \ge 1$ for $n = 1, 2, 3 \dots$. But then $f\left(\frac{x_i + x_{i+1}}{2}\right) \geq f(a) + i$, for $i = 1, 2, 3 \dots$,

making f unbdd. Thus the set of pts in $[a,b]$ where $ocs_x(f) \ge 1$ is finite. Likewise, for any $k \in \mathbb{N}$, the set D_k where $ocs_x(f) \ge 1/k$ is finite.

Thus the set $D = \cup D_k$ of points where $ocs_x(f) \neq 0$ is at most countable and hence of measure zero. ∎

27 If $f, g \in \mathcal{R}$ then $f \cdot g \in \mathcal{R}$.

Pf $D(f \cdot g) = D(f) \cup D(g)$.

28 If $f : [a,b] \to [c,d]$ is in $\mathcal{R}$ and $\phi : [c,d] \to \mathbb{R}$ is continuous, then $\phi \circ f : [a,b] \to \mathbb{R}$ is in $\mathcal{R}$.

Pf Any discontinuous point for $\phi \circ f$ must also be a " " " f. Since a subset of a zero set is a zero set, $\phi \circ f \in \mathcal{R}$. ∎

Note If f and ϕ are merely both R.I. it need not be the case that $f \circ \phi$ is. See exercise 33. See also Corollary 32 & 33 ~~below~~ in text.

29 $f \in \mathcal{R} \Rightarrow |f| \in \mathcal{R}$. See also exercise 52.

Pf See textbook.

30 If $a < b < c$ and $f : [a,c] \to \mathbb{R}$ is R.I then the integrals below exist and satify

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$$

Pf See textbook.

31 If $f : [a,b] \to [0,M]$ is R.I. and $\int_a^b f(x)\,dx = 0$, then $f(x) = 0$, a.e. See textbook for proof.

34 $\boxed{\text{The FTC.}}$ Let $f:[a,b]\to\mathbb{R}$ be Riemann integrable. Define its indefinite integral to be

$$F(x) = \int_a^x f(t)\,dt.$$

Then F exists and is continuous on $[a,b]$ and if $x \in [a,b]$ is a point where f is continuous, then

$$F'(x) = f(x).$$

Rmk Before you study the proof, make sure you understand the graphical "proof."

Pf $F(x)$ exists for $x \in [a,b]$ by Corollary 30. We know f is bdd so we let $M>0$ be s.t. $|f(x)| \geq M$ over $[a,b]$. Thus we have,

$$|F(y)-F(x)| = \left|\int_a^y f(t)\,dt - \int_a^x f(t)\,dt\right| = \left|\int_x^y f(t)\,dt\right| \leq M|y-x|.$$

Let $\varepsilon>0$. Let $\delta = \frac{\varepsilon}{M+1}$ Now $|y-x|<\delta$ implies

$$|F(y)-F(x)| \leq M\delta = \frac{M}{M+1}\varepsilon < \varepsilon.$$

Thus, F is continuous.

Now assume f is continuous at some $x \in [a,b]$. We compute,

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{1}{h} \int_{x}^{x+h} f(t)\,dt.$$

We want to show this limit is $f(x)$. Fix h for the moment and define

$$m(x,h) = \inf \{ f(s) : |s - x| \leq |h| \}, \text{ and}$$

$$M(x,h) = \sup \{ f(s) : |s - x| \leq |h| \}.$$

Each of these converges to $f(x)$ as $h \to 0$ because f is continuous at x. This will allow us to use the Squeeze Theorem.

$$m(x,h) = \frac{1}{h} \int_{x}^{x+h} m(x,h)\,dt \leq \frac{1}{h} \int_{x}^{x+h} f(t)\,dt \leq \frac{1}{h} \int_{x}^{x+h} M(x,h)\,dt = M(x,h).$$

By the Squeeze Theorem

$$\frac{1}{h} \int_{x}^{x+h} f(t)\,dt \to 0 \quad \text{as} \quad h \to 0.$$

35 **Corollary.** The derivative of an indefinite Riemann integral exists a.e. (almost everywhere) and equals the integrand a.e.

Pf RLT + FTC. See textbook.

Def If $F'(x)=f(x)$ on $[a,b]$ (everywhere) then we say that $F(x)$ is an antiderivative of f.

36 **Corollary.** Every continuous function has an antiderivative

Pf See textbook.

Ex (Exercise 40) Let $f(x)=\begin{cases} 0 & x\le 0 \\ \sin(\frac{\pi}{x}) & x>0 \end{cases}$ and $g(x)=\begin{cases} 0 & x\le 0 \\ 1 & x>0 \end{cases}$.

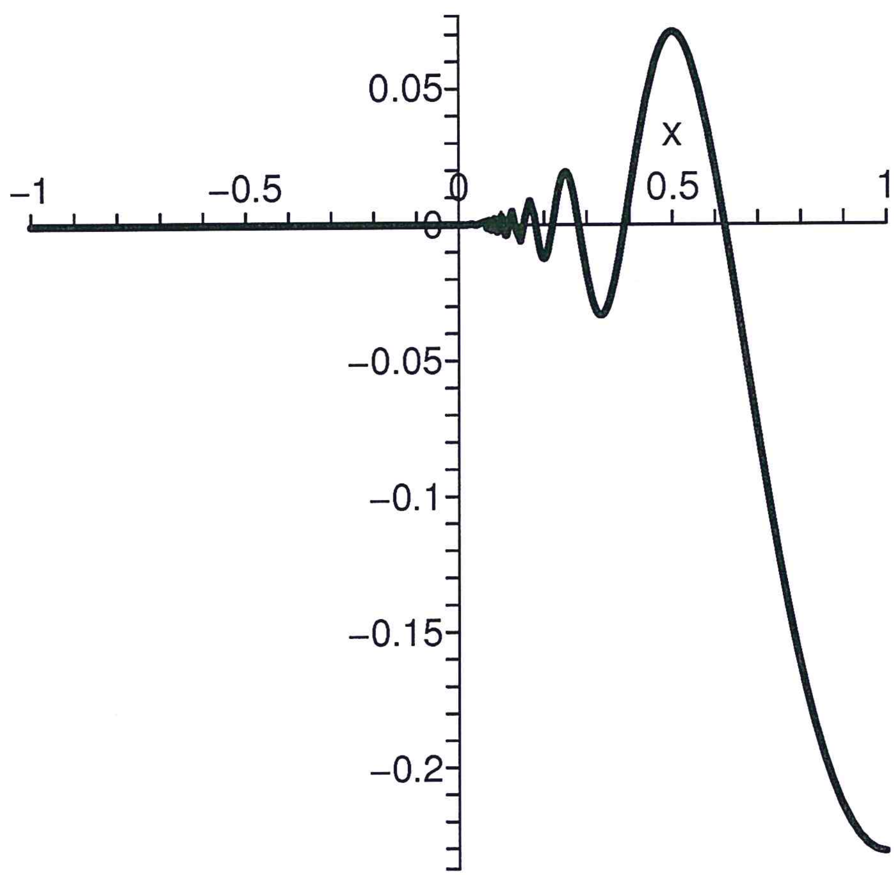
Then $f(x)$ has an antiderivative but $g(x)$ does not. (It is understood that we are using an interval that contains 0 in its interior, say $[-1,1]$.)

This is easy to see for g. Show $G(x)=\begin{cases} C_1 & x\le 0 \\ x+C_2 & x>0 \end{cases}$ if $G'(x)=g(x)$ $\forall x\neq 0$. To be cont. we need $C_1=C_2$. But ~~there~~ is no way $G'(0)$ can exist.

To get an idea for f integrate it numerically.

```
> with(plots); #Loads various plotting functions.

> part2:=plot(int(sin(Pi/t), t=0..x),x=0..1, color=black,thickness=2):
> part1:=plot(0,x=-1..0, color=black,thickness=2):
> display(part1,part2);
```



So, it is at least plausable that for

$F(x) = \int_{-1}^{x} f(t)\,dt$ we have that $F'(x)$ exists

and equals $f(x)$ on $[-1, 1]$. Try to

proof this.

Aside. The integral $\int \sin(\frac{1}{x})\,dx$ cannot be done is closed form. But you can do this much:

$$\int \sin\left(\tfrac{1}{x}\right) dx = -\int \frac{\sin(w)}{w^2}\,dw$$

Subst. $\rightarrow \left(w = \frac{1}{x} \quad dw = \frac{-1}{x^2}\,dx \right)$

$$= \frac{\sin(w)}{w} - \int \frac{\cos(w)}{w}\,dw = x \sin\left(\tfrac{1}{x}\right) - Ci\left(\tfrac{1}{x}\right).$$

Int by Parts $\rightarrow \left(\begin{array}{ll} u = \sin(w) & du = \cos(w)\,dw \\ dv = -\frac{1}{w^2}\,dw & v = \frac{1}{w} \end{array} \right)$

The symbol Ci is called the cosine integral.

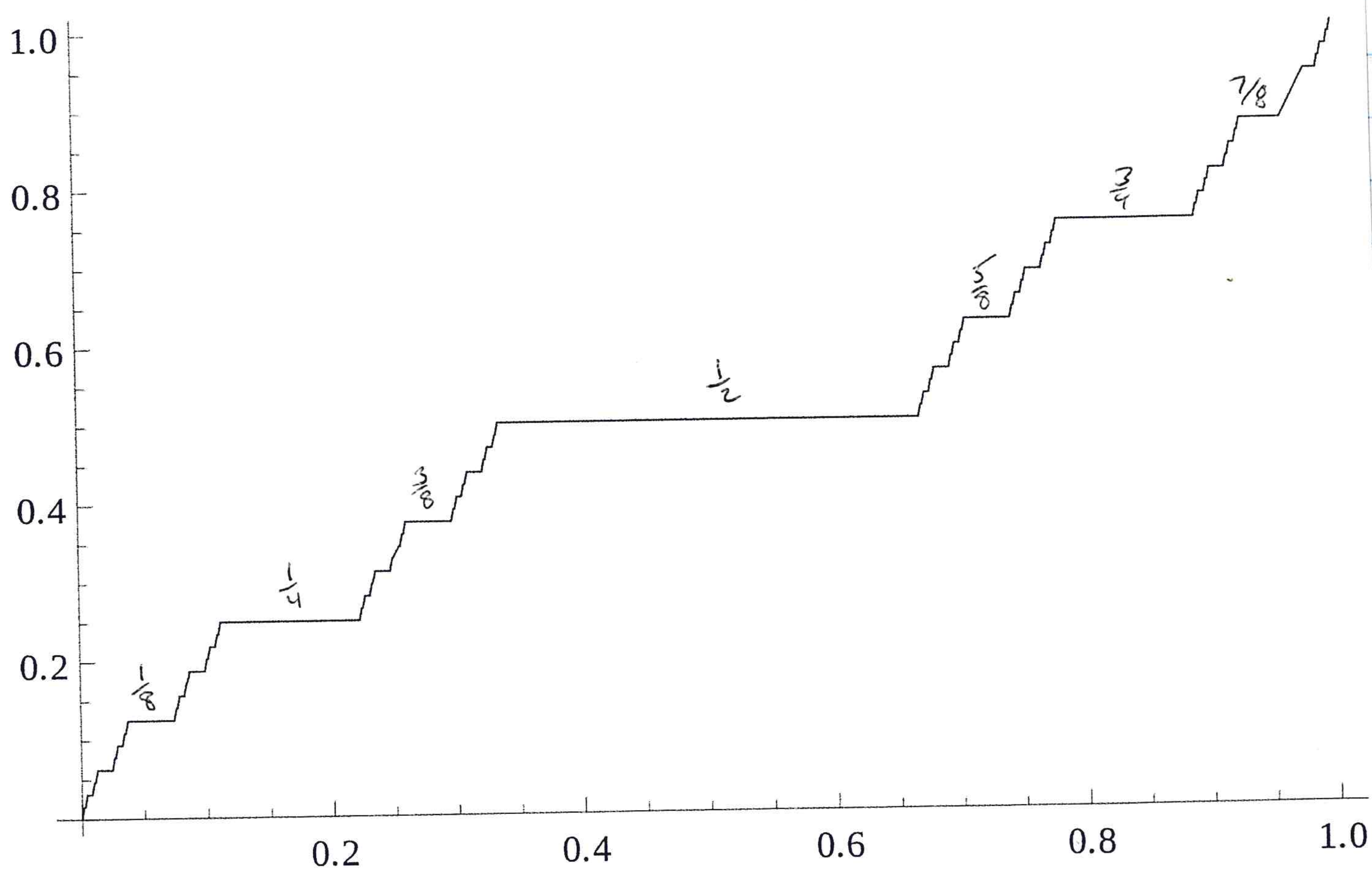
$$Ci(x) = \int \frac{\cos(x)}{x}\,dx.$$

37 **The Antiderivative Theorem** Suppose $f:[a,b]\to\mathbb{R}$ is Riemann integrable and that $G:[a,b]\to\mathbb{R}$ is an antiderivative of f. Then

$$G(x) = \int_a^x f(t)\,dt + C.$$

Pf See textbook.

Ex The Devil's staircase. We will define a cont. func. $H:[0,1]\to\mathbb{R}$ s.t. $H'(x)=0$ a.e. but H is not constant. Here is a graph of $H(x)$ from wikipedia.



Recall we had defined a cont. map from the middle thirds Cantor set C onto $[0,1]$:

$$H(x) = \sum_{i=1}^{\infty} \frac{x_i/2}{2^i},$$

where $(x_1 x_2 x_3 \cdots)$ is the base three expansion of $x \in C$ express so that $x_i = 0$ or 2 for all i. If we let $y = H(x)$ and $y_i = x_i/2$, $i = 1, 2, \cdots$, the $(y_1 y_2 y_3 \cdots)$ is the based expansion of $y \in [0,1]$.

To extend H to $[0,1]$, recall that H on the end points of any deleted interval took the same value. So, we define H to be this value on each removed interval. Thus,

$$H(x) = \tfrac{1}{2} \quad \text{for } x \in [\tfrac{1}{3}, \tfrac{2}{3}].$$
$$H(x) = \tfrac{1}{4} \quad \text{for } x \in [\tfrac{1}{9}, \tfrac{2}{9}]$$
$$H(x) = \tfrac{3}{4} \quad \text{for } x \in [\tfrac{7}{9}, \tfrac{8}{9}]$$
$$H(x) = \tfrac{1}{8} \quad \text{for } x \in [\tfrac{1}{27}, \tfrac{2}{27}]$$

$$H(x) = \tfrac{3}{8} \quad \text{for } x \in [\tfrac{7}{27}, \tfrac{8}{27}]$$
$$H(x) = \tfrac{5}{8} \quad \text{for } x \in [\tfrac{19}{27}, \tfrac{20}{27}]$$
$$H(x) = \tfrac{7}{8} \quad \text{for } x \in [\tfrac{25}{27}, \tfrac{26}{27}]:$$

etc. etc.

Here is a way to formalize this. Let $y = H(x)$ and write $x = (.x_1 x_2 x_3 \dots)_3$, $y = (y_1 y_2 y_3 \cdots)_2$. Then

$$y_i = \begin{cases} x_i/2 & \text{if } x_i = 0 \text{ or } 2 \text{ and } x_j \neq 1 \text{ for } j < i. \\ 1 & \text{if } x_i = 1 \text{ and } x_j \neq 1 \text{ for } j < i. \\ 0 & \text{if } \exists\, j < i \text{ s.t. } x_j = 1. \end{cases}$$

In words, once $x_i = 1$ all $y_{k>i} = 0$, the first time $x_i = 1$, $y_i = 1$, and otherwise $y_i = x_i/2$.

See the textbook for the proofs that H is well defined and continuous. Well defined means if $(.x_1 x_2 x_3 \dots)_3 = (.x_1' x_2' x_3' \dots)_3$ then H gives the same value. For example,

$$H(.020222\dots)_3 = (.010111\dots)_2 = (.01100\dots)_2 = \frac{3}{8}$$

and

$$H(.021000\dots) = (.01100\dots)_2 = \frac{3}{8}.$$

You can read about the integral definition of $\ln(x)$, int. by subst., int by parts, and improper integrals on your own.

I do want to remark that the text's statement of the int. by subst. is a bit stronger than in calculus books. This is why the proof given goes back to Riemann sums instead just using the chain rule backwards.

Calc: $\int_c^d F'(g(x))\, g'(x)\, dx = \int_c^d [F(g(x))]'\, dx$

$$= F(g(d)) - F(g(c)).$$

Text: $\int_a^b f(g(y))\, dy = \int_c^d f(g(x))\, g'(x)\, dx$.

where $f \in \mathcal{R}([a,b])$ $g : [c,d] \to [a,b]$ is increasing.

Ex: $\int_0^2 \cos(\underset{u}{\underset{\shortparallel}{x^3}})\, 3x^2\, dx = \int_0^8 \cos(u)\, du = \sin(8) - \sin(0)$.

For teaching about improper integrals I have the class compute $\int_{-1}^1 \frac{1}{x^2}\, dx$. They get $\underline{-2}$ and do not see anything wrong with this!