Ch. 3
Sec. 3

Series

Unlike baseball, a series is different from a sequence.

$$\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} \sum_{n=1}^{k} a_n.$$

Applying the Cauchy criteria for convergence to a series gives

$$\sum_{n=1}^{\infty} a_n \text{ convergences iff } \forall \varepsilon>0\ \exists N>0$$

$$\text{s.t. } p,k \geq N \Rightarrow \left| \sum_{n=p}^{k} a_n \right| < \varepsilon.$$

There are many standard tests for series convergence that you are almost surely familar with.

For a geometric series, $\sum_{n=0}^{\infty} r^n$, convergences to $\frac{1}{1-r}$ for $|r|<1$ and diverges otherwise.

The comparison test: If $\sum b_n$ converges and all $|a_n| \leq b_n$, $\sum a_n$ converges. We say $\sum b_n$ dominates $\sum a_n$.

The integral test is basically the same idea.

Not mentioned is the Limit Comparison Test: If $\sum \frac{a_n}{b_n} = c$ (positive series), then either $\sum a$ $\sum b$ both converge or both diverge.

The integral test can be used to get the p-series test.

The n^{th} term test: If $\sum a_n$ converges, then $a_n \to 0$. If $\sum a_n$ is an alternating series it converges iff $a_n \to 0$.

The limit comparison test is not in your text but is in many calc. textbooks: Assume (a_n) and (b_n) are positive. If $\lim \frac{a_n}{b_n}$ exists (and is finite) then either $\sum a_n$ and $\sum b_n$ both converges or both diverge.

Def If $\sum |a_n|$ converges, it follows that $\sum a_n$ converges and we say $\sum a_n$ ~~conv~~ absolutely convergent. If $\sum a_n$ converges but $\sum |a_n|$ does not we say $\sum a_n$ is conditionally convergent.

Ex $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is cond. conv.

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is abs. conv.

Thm (The Ratio Test) Let $\sum a_k$ be an infinite series and assume no a_k equals zero. Let

$$\lambda = \liminf_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| \qquad \rho = \limsup \left|\frac{a_{k+1}}{a_k}\right|.$$

If $\rho < 1$ then $\sum a_k$ and $\sum |a_k|$ converge, i.e. we have abs. conv.

If $\lambda > 1$ then $\sum a_n$ diverges.

Pf Suppose $\rho < 1$. Choose $\beta \in (\rho, 1)$. Then

$$\exists N \in \mathbb{N} \text{ s.t. } k \geq N \Rightarrow \left|\frac{a_{k+1}}{a_k}\right| < \beta$$

Then $|a_{k+1}| < \beta|a_k| < \beta^2|a_{k-1}| < \beta^3|a_{k-2}| < \cdots < \beta^{k-N}|a_N|$,

where we have $k \geq N$.

Thus, $|a_k| < \beta^{k-N}|a_N|$. Let $C = \beta^{-N}|a_N|$,

a constant independent of k. Thus for $k \geq N$

$$|a_k| < \beta^k C$$

Thus $\sum_{k=N}^{\infty} |a_k|$ converges by comparison with the geom. series $\sum_{k=N}^{\infty} C\beta^k$. Thus $\sum |a_k|$ and $\sum a_k$ conv.

If $\lambda > 1$ choose $\beta \in (1, \lambda)$. By a similar argument $|a_k| > \beta^k / C$ for k large enough.

Hence $a_k \not\to 0$. Thus $\sum a_k$ diverges.

Thm (Root Test) Let $\sum a_k$ be an infinite series. Let $\alpha = \limsup_{k \to \infty} \sqrt[k]{|a_k|}$. This is called the

exponential growth rate. If $\alpha < 1$ the series converges. If $\alpha > 1$ the series diverges. If $\alpha = 1$, no conclusion can be drawn.

Pf See textbook.

Rmks The root test is strictly stronger than the ratio test, although the ratio test is often easier to use.

Both proofs are base on a comparison with a geometric series. It can be useful for calc students to at least see this as it makes it easier to see why the "break point" is 1.

We will deal with series of functions in Ch. 4. For now we just note:

Thm Let $\sum c_k x^k$ be a power series. $\exists R \in [0, \infty]$ s.t. $|x| < R \Rightarrow \sum c_k x^k$ converges while
$|x| > R \Rightarrow \sum c_k x^k$ diverges.

In fact

$$R = \frac{1}{\limsup_{k \to \infty} \sqrt[k]{|a_k|}}$$

Pf Use root test, see textbook.

R

Def ~~The reason~~ It is called the radius of convergence. The reason for this name is that all this works in the complex plane and the convergence regions are disks. I find it useful to tell calculus students this.

Def Let $\sum a_n$ be an infinite series. Let $\beta: \mathbb{N} \to \mathbb{N}$ be one-to-one and onto. Then $\sum a_{\beta(n)}$ is called a rearrangement of $\sum a_n$.

If $\sum a_n$ is abs. conv. rearrangements do not change the result.

But if $\sum a_n$ is cond. conv. and $X \in \mathbb{R} \cup \{-\infty, \infty\}$ then $\exists\, \beta: \mathbb{N} \to \mathbb{N}$ s.t. $\sum_{n=1}^{\infty} a_{\beta(n)} = X$.

For a proof of the second fact, see

Riemann's Rearrangement Theorem by Stewart Galanor, Mathematics Teacher, Nov. 1987, Vol 80, Num. 8 pp 675-681.

We will do the proof of the first next.

Thm The Rearrangement Thm. Let $\sum a_n$ be abs. conv. Let $\beta : \mathbb{N} \to \mathbb{N}$ be one-to-one and onto. Then $\sum a_{\beta(n)}$ is abs. conv. and has the same value as $\sum a_n$.

Pf [From The Elements of Real Analysis, 2nd edition, by R. G. Bartle, page 292.]

Let $y_n = x_{\beta(n)}$. Let $s_p = \sum_{n=1}^{p} x_n$ and $t_q = \sum_{n=1}^{q} y_n$.

Let K be an upper bound for $\left\{ \sum_{n=1}^{p} |x_n| \right\}_{p=1}^{\infty}$.

Take q as given for now and let $p = \max\{\beta^{-1}(n) \mid n = 1, \dots, q\}$. Then

$$\sum_{n=1}^{q} |y_n| \le \sum_{n=1}^{p} |x_n| \le K.$$

Since $\left\{ \sum_{n=1}^{q} |y_n| \right\}_{q=1}^{\infty}$ has an upper bound and is increasing it has a limit; that is $\sum_{n=1}^{\infty} |y_n|$ converges.

It follows the $\sum y_n$ converges too.

Now we show $\sum^{\infty} y_n = \sum^{\infty} x_n$. Let $y = \sum^{\infty} y_n$ and $x = \sum^{\infty} x_n$.

Let $\varepsilon > 0$. Let N be s.t. if $q > p \geq N$ we have

$$|x - s_p| < \varepsilon \quad \text{and} \quad \sum_{k=p+1}^{q} |x_k| < \varepsilon.$$

(The second ineq comes from the Cauchy condition.) Fix p for now. Choose $r \in \mathbb{N}$ s.t.

$$|y - t_r| < \varepsilon \quad \text{and}$$

each of $x_1, x_2, x_3, \dots, x_p$ occurs in the terms for t_r. Now choose $q > p$ s.t. every y_n in t_r is also in s_q.

Thus,

$$|x - y| \leq |x - s_q| + |s_q - t_r| + |t_r - y| < \varepsilon + \sum_{k=p+1}^{q} |x_k| + \varepsilon < 3\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $x = y$.

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