Ch 4 Sec 1

Function Spaces

Uniform vs Pointwise convergence.

Def For each $n \in \mathbb{N}$ let $f_n : [a,b] \to \mathbb{R}$, and $f : [a,b] \to \mathbb{R}$.

If for each $x \in [a,b]$ the $\lim_{n\to\infty} f_n(x) = f(x)$ then we say f is the pointwise limit of (f_n).

If $\forall \varepsilon > 0 \; \exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow |f_n(x) - f(x)| < \varepsilon$ $\forall x \in [a,b]$, then we say f_n converges uniformly to f on $[a,b]$.

Notation $f_n \to f$ means pointwise convergence.

$f_n \rightrightarrows f$ means uniform convergence.

Ex Let $f_n : [0,1] \to \mathbb{R}$ be given by $f_n(x) = x^n$.
Let $f : [0,1] \to \mathbb{R}$ be

$$f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1. \end{cases}$$

Then $f_n \to f$ but $f_n \not\rightrightarrows f$.

You should check this.

Thm If $f_n \rightrightarrows f$ and each f_n is continuous at x_0 then f is continuous at x_0 also.

Pf Let $f_n : [a,b] \to \mathbb{R}$ be continuous at $x_0 \in [a,b]$. Assume $f_n \rightrightarrows f$ on $[a,b]$.

Let $\varepsilon > 0$. $\exists N \in \mathbb{N}$ s.t. $n \geq N$ implies

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3}, \ \forall x \in [a,b].$$

$\exists \delta > 0$ s.t.

$$|x - x_0| < \delta \Rightarrow |f_N(x) - f_N(x_0)| < \frac{\varepsilon}{3}.$$

Thus, when $|x - x_0| < \delta$,

$$|f(x) - f(x_0)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| < \varepsilon$$

$\boxed{\%}$

Def Let $C_b = C_b([a,b], \mathbb{R})$ = all bdd functions $[a,b] \to \mathbb{R}$.

The sup norm on C_b is given by

$$\|f\| = \sup\{|f(x)| : x \in [a,b]\}.$$

You can check that norm cirteria are satisfied:

$$\|f\| \geq 0 \quad , \|f\| = 0 \text{ iff } f(x) = 0, \forall x.$$

$$\|cf\| = |c|\,\|f\|$$

$$\|f+g\| \leq \|f\| + \|g\|.$$

This gives us a metric on C_b,

$$d(f,g) = \sup\{|f(x) - g(x)|\} = \|f-g\|.$$

Thm Convergence w.r.t. the sup metric is equivalent to uniform convergence.

Pf Easy. See textbook, Thm 2, pg 216.

Thm C_b with the sup metric is a complete metric sp.

Pf Let (f_n) be a Cauchy seq in C_b. First, we will show that it has a pointwise limit. Then we will show that the convergence is uniform. Finally, we will show that this limit is bdd, and hence in C_b.

Let $\varepsilon > 0$. Let $N \in \mathbb{N}$ be s.t. (Let $x_0 \in [a,b]$.)

$$m, n \geq N \Rightarrow \underbrace{d}_{\text{sup metric.}}(f_n, f_m) < \varepsilon.$$

Thus,

$$|f_n(x_0) - f_m(x_0)| \leq \sup_{x \in [a,b]} \{|f_n(x) - f_m(x)|\} = d(f_n, f_m) < \varepsilon.$$

Therefore, $\lim_{n \to \infty} f_n(x_0)$ exist for any $x_0 \in [a,b]$.

Define $\boxed{f(x) = \lim_{n \to \infty} f_n(x)}$ for each $x \in [a,b]$.

Now we show that the convergence is in fact uniform. Let $\varepsilon > 0$. Let N_1 be s.t. $m, n \geq N_1$ implies

$$d(f_m, f_n) < \varepsilon/2.$$

Let $x \in [a,b]$. Let N_2 be s.t. $n \geq N_2$ implies

$$|f_n(x) - f(x)| < \varepsilon/2.$$

Let $n_1 \geq N_1$ and $n_2 \geq \max\{N_1, N_2\}$. Then

$$|f_{n_1}(x) - f(x)| \leq |f_{n_1}(x) - f_{n_2}(x)| + |f_{n_2}(x) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

Since N_1 is independent of x, the convergence is uniform.

Lastly, we need to show f is bdd. $\exists N \in \mathbb{N}$ s.t. $\|f_N - f\| < 1$. Thus,

$$|f_N(x) - f(x)| < 1 \quad \forall x \in [a,b].$$

Since $f_N \in C_b$ $\exists M \geq 0$ s.t. $|f_N(x)| < M$ $\forall x \in [a,b]$. But now we know that $|f(x)| < M+1$. Thus $f \in C_b$.

$\square$

Corollary $C^0([a,b], \mathbb{R})$ is a closed, complete subspace of $C_b([a,b], \mathbb{R})$.

Pf Two lines. Figure it out.

Next we look at series of functions: $\sum_{k=0}^{\infty} f_k(x)$.

Thm (The Weierstrass M-test) For $k=0,1,2,3,\dots$ let $f_k(x) \in C_b$ and suppose $\|f_k\| \le M_k$. If $\sum M_k$ convergence, then $\sum f_k(x)$ converges uniformly.

Pf Let $F_n(x) = \sum_{k=0}^{n} f_k(x)$, $\forall x \in [a,b]$. Let $n > m$. Then

$$d(F_n, F_m) \le d(F_n, F_{n-1}) + d(F_{n-1}, F_{n-2}) + \dots + d(F_{m+1}, F_m)$$

$$= \|F_n - F_{n-1}\| + \dots + \|F_{m+1} - F_m\| = \|f_n\| + \dots + \|f_m\| \le \sum_{k=m+1}^{n} M_k .$$

Let $\varepsilon > 0$. Then $\exists N \ge 0$ s.t. $n > m \ge N \Rightarrow \sum_{k=m+1}^{n} M_k < \varepsilon$.

Thus $n, m \ge N \Rightarrow d(F_n, F_m) < 0$. Thus $(F_n) = \left(\sum_{k=0}^{n} f_k(x)\right)$ converges uniformly.

Now we present some basic results involving integerbility, and differentiability.

Thm Let $f_n : [a,b] \to \mathbb{R}$ be R.I. (hence $f_n \in C_b$). Suppose $f_n \rightrightarrows f$. Then

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \text{unif}\lim_{n\to\infty} f_n(x)\,dx.$$

Pf By the RLT each $f_n \in C_b$. Let Z_n be the set of points in $[a,b]$ where f_n is not continuous. By the RLT Z_n is a zero set. Then $Z = \bigcup Z_n$ is a zero set. Each f_n is cont. $\forall x \in [a,b]-Z$. Thus, f is cont on $[a,b]-Z$. We know that $f \in C_b$. Thus $f \in \mathcal{R}$.

To finish, we have

$$\left|\int_a^b f(x)\,dx - \int_a^b f_n(x)\,dx\right| = \left|\int_a^b f(x) - f_n(x)\,dx\right|$$

$$\leq \int_a^b |f(x) - f_n(x)|\,dx \leq (b-a)\, d(f, f_n) \to 0.$$

Thus, $\int_c^b f_n(x)\,dx \longrightarrow \int_a^b f(x)\,dx.$

Cor If $f_n \in \mathcal{R}$, $n=1,2,3\ldots$ and $f_n \rightrightarrows f$, then

$$\int_a^x f_n(t)\,dt \rightrightarrows \int_a^x f(t)\,dt.$$

Cor If $\sum_{k=0}^{\infty} f_k \rightrightarrows F$, each $f_k \in \mathcal{R}$, then

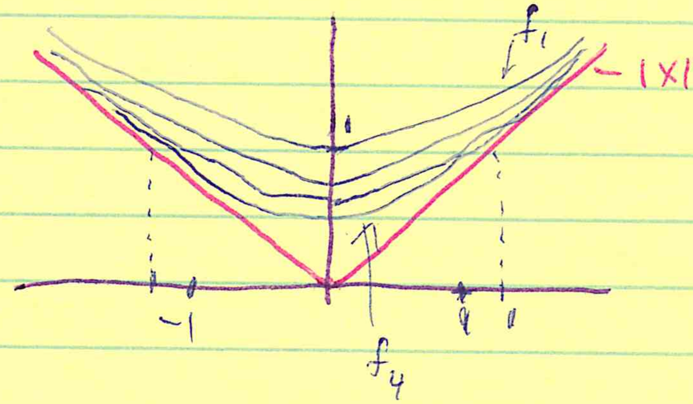
$$\sum_{k=0}^{\infty} \int_a^b f_k(x)\,dx = \int_a^b \sum_{k=0}^{\infty} f_k(x)\,dx.$$

See textbook for proofs of the corollaries, Corollary 7, pg 218 and Thm 8 pg 219.

The situation for differentiability is different

Ex Let $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$ for $x \in [-1,1]$. Each f_n is differentiable. The $\lim_{n\to\infty} f_n(x) = \sqrt{x^2} = |x|$,

which is not diff. at $x=0$. You can check that the convergence is uniform.



But, we do have the following.

Thm Suppose $f_n \rightrightarrows f$ and $f_n' \rightrightarrows g$. Then $f' = g$.

Pf Let

Pick $x \in [a,b]$.

$$\phi_n(t) = \begin{cases} \dfrac{f_n(t) - f_n(x)}{t - x} & t \neq x \\ f_n'(x) & t = x; \end{cases}$$

$$\phi(t) = \begin{cases} \dfrac{f(t) - f(x)}{t - x} & t \neq x \\ g(x) & t = x. \end{cases}$$

Each $\phi_n(t)$ is cont. (Check limit as $t \to x$.)

Clearly $\phi_n(x) \to \phi(x)$, pointwise. We claim $\phi_n \rightrightarrows \phi$.

$\forall m,n$ the MVT gives

$$\phi_m(t) - \phi_n(t) = \frac{[f_m(t) - f_n(t)] - [f_m(x) - f_n(x)]}{t - x} = f_m'(\theta) - f_n'(\theta)$$

for some θ between t and x. Since $f_n' \rightrightarrows g$, $f_m' - f_n' \rightrightarrows g - g = 0$ as $m,n \to \infty$, (This means $\forall \varepsilon > 0$ $\exists N$ s.t. $m,n \geq N \to d(f_m', f_n') < \varepsilon$.) it follows that

the seq. (ϕ_n) is Cauchy in C^0. Since C^0 is complete $\exists \psi \in C^0$ s.t. $\phi_n \rightrightarrows \psi$.

Since $\phi_n(x) \to \phi(x)$, pointwise, we know $\psi(x) = \phi(x)$.

Since ψ is cont (check $\lim_{t \to x} \psi(t) = \psi(x)$), we have

$f'(x) = g(x)$ as desired.

$\boxed{\$}$

Cor Let $\sum f_k(x)$ be a uniformly convergent series of diff. functions, and suppose $\sum f_k'(x)$ also converges uniformly, then

$$\left(\sum_{k=0}^{\infty} f_k(x) \right)' = \sum_{k=0}^{\infty} f_k'(x)$$

Pf See textbook, (Thm 10, pg 220).