Ch 4
Sec 3

Compactness and Equicontinuity in C^0

In C^0 closed & bdd does not imply compact!
Even the closed unit ball is not compact.

Ex Let $B = \{ f \in C^0([0,1], \mathbb{R}) \mid \|f\| \leq 1 \}$.

Then the seq $(x^n) \subset B$; in fact $\|x^n\| = 1$.
But there is no convergent subseq (with limit in C^0, $\rightarrow \, i \notin C^0$).

Q We know C^0 is complete. Thus (x^n) must not be Cauchy. Show this directly.

Q Think about the unit ball in $C_b([0,1], \mathbb{R})$.

Note The textbook points out that ~~the~~ closed unit ball of a vector space is compact iff the space is finite dimensional.

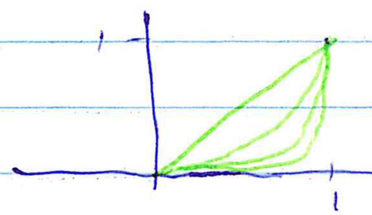
Q Can we come up with an extra condition on a closed bdd set that would make it compact in C^0?

Def Let $\mathcal{E} \subset C^0$. Then $\mathcal{E}$ is equicontinuous if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall f \in \mathcal{E}$,

$$|s-t| < \delta \Rightarrow |f(s) - f(t)| < \varepsilon.$$

The idea is that the functions are "equally continuous" or that their "stretchiness" is uniform. δ is independent of $f \in \mathcal{E}$.

Show that $\{x^n\}$ on $[0,1]$ is not equicontinuous.



A $\delta > 0$ that works for x^3 may not work for x^{100}.

Thm The Arzelà-Ascoli Thm Any bdd equicont. seq (f_n) in $C^0([a,b], \mathbb{R})$ has a uniformly convergent sub seq.

Pf Since the seq is bdd $\exists M > 0$ s.t. $\|f_n\| < M, \forall n$.

Let $D = \{d_1, d_2, d_3, d_4, \dots\}$ be a countable dense subset of $[a,b]$. Then for $i = 1, 2, 3 \dots$, the seq $(f_n(d_i))_{n=1}^{\infty}$ is bdd seq of real numbers.

Now, $\exists$ a subseq of $(f_n(d_1))_{n=1}^{\infty}$ that converges:

Suppose, $f_{n_k}(d_1) \to y_1$ as $k \to \infty$.

Next consider $(f_{n_k}(d_2))_{k=1}^{\infty}$. It is a bdd seq of real numbers and hence has a convergent subseq. We could call it $(f_{n_{k_j}}(d_2))_{j=1}^{\infty}$, but the notation gets clumsy, especially since we are going to do this over and over. Call the first subseq $(f_{1,k}(d_1))$ and the second $(f_{2,k}(d_2))$.

Now $(f_{2,k}(d_3))$ is a bdd seq of real numbers. Thus it has a conv. subseq., $(f_{3,k}(d_3))$. Continue in this way. Thus for any m we have

$f_{m,k}$ is a subseq of $f_{m-1,k}$ and

$$f_{m,k}(d_m) \to y_m, \text{ as } k \to \infty.$$

$$\text{If } j \le m, \ f_{m,k}(d_j) \to y_j \text{ as } k \to \infty.$$

Let $g_m = f_{m,m}$, $m = 1, 2, \cdots$, the "diagonal seq".

For any i we claim $g_m(d_i) \to y_i$. Why? Eventually, $m \geq i$. Then $(g_m(d_i))_{m \geq i}$ is a subseq of

$(f_{i,k}(d_i))$. Thus it has the same limit, y_i.

Thus $(g_m(x))$ converges for all $x \in D$.

We need to show $(g_m(x))$ converges $\forall x \in [a,b]$ and that the conv. is unif. We will do both by showing that (g_m) is a Cauchy seq in C^0.

Let $\varepsilon > 0$. By assumption (f_n) is equicont. Thus, $\exists \delta > 0$ s.t. $\forall s, t \in [a,b]$,

$$|s-t| < \delta \Rightarrow |g_m(s) - g_m(t)| < \frac{\varepsilon}{3}.$$

Cover $[a,b]$ with $\{(d_i - \delta, d_i + \delta) \mid i = 1, 2, 3 \ldots\}$. It has a finite subcover $\{(d_{i_k} - \delta, d_{i_k} + \delta) \mid k = 1, \ldots n\}$. Let $J = \max\{i_k\}$. We will work with $\{d_1, d_2, \ldots, d_J\}$.

$\exists N$ s.t. for $p, q \geq N$ and any $j \leq J$, we have

$$|g_p(d_j) - g_q(d_j)| < \frac{\varepsilon}{3}.$$

Given $p, q \geq N$ and $x \in [a,b]$, $\exists\, j \in J$ s.t. $|d_j - x| < \delta$. Thus

$$|g_p(x) - g_q(x)| \leq |g_p(x) - g_p(d_j)| + |g_p(d_j) - g_q(d_j)| + |g_q(d_j) - g_q(x)|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Therefore (g_m) is Cauchy in C^0, and so converges uniformly to a member of C^0. It is the desired seq sub of (f_n).

Cor Let $f_n : [a,b] \to \mathbb{R}$, $n = 1, 2, 3, \ldots$, be differentiable (and hence cont. and individually bdd), and suppose $\exists\, M > 0$ s.t. $|f_n'(x)| \leq M$ $\forall x \in [a,b], n \in \mathbb{N}$. If $\exists\, x_0 \in [a,b]$ s.t. $|f_n(x_0)|$ is bdd for $n \in \mathbb{N}$, then (f_n) has a sub seq that converges uniformly on $[a,b]$.

We give two counterexamples when some of the hypotheses are weakened. Then we do the proof.

C. Ex. 1 $\{x+n : [0,1] \to \mathbb{R} \mid n = 1,2,3\ldots\}$ satisfies all the conditions but the last and clearly there is no conv. subseq.

C. Ex. 2 $\{nx : [0,1] \to \mathbb{R} \mid n = 1,2,3,\ldots\}$ satisfies the last condition (let $x_0 = 0$) but the derivatives are unbdd. Again, no conv. subseq. exists.

Pf Let $\varepsilon > 0$ be given and let $\delta = \frac{\varepsilon}{M+1}$. For $s<t$, both in $[a,b]$, the MVT says $\exists\, \theta \in (s,t)$ s.t.

$$\frac{f_n(t)-f_n(s)}{t-s} = f_n'(\theta).$$

Thus, if $|t-s|<\delta$, then $|f_n(t)-f_n(s)| < (t-s)|f_n'(\theta)| < \delta M < \varepsilon$.

Thus, (f_n) is equicontinuous.

Let c be a bd for $\{|f_n(x_0)|\}_{n=1}^{\infty}$. Then

$$|f_n(x)| \leq |f_n(x)-f_n(x_0)| + |f_n(x_0)| \leq |x-x_0|M + c \leq (b-a)M+c.$$

Thus (f_n) is uniformly bdd. By A-A, (f_n) has a unif. conv. subseq.

Thm The Heine-Borel Thm in a Function Space.
A subset $\mathcal{E} \subset C^0$ is compact iff it is closed, bdd and equicontinuous.

Before starting the proof we review "total boundedness" from Ch 2, pgs 103-104.

Def A subset A of a metric space M is **totally bounded** if $\forall \varepsilon > 0$, $\exists$ a finite subset of A, $\{a_1, a_2, \dots a_n\}$ s.t. $\{ B(a_i, \varepsilon) \mid i = 1, 2, \dots n \}$ covers A.

Thm The Generalized Heine-Borel Thm. A subset of a complete metric space is compact iff it is closed and totally bounded.

Pf See textbook, pgs 103-104.

Pf (HB Thm in C^0)
The A.A Thm gives one direction. Suppose $\mathcal{E}$ is closed, bdd, and eq. cont. If (f_n) is a seq in $\mathcal{E}$ some subseq (f_{n_k}) conv. unif. to a limit, by AA. The limit lies in $\mathcal{E}$ since $\mathcal{E}$ is closed. Thus $\mathcal{E}$ is seq. compact.

For the other direction assume $\mathcal{E}$ is compact.
By the Gen. HB Thm it is closed and totally bdd.

Let $\varepsilon > 0$, and pick any $f \in \mathcal{E}$. $\exists f_1, \dots f_n \in \mathcal{E}$ s.t. $\{B(f_k, \frac{\varepsilon}{3}) \mid k = 1, \dots n\}$ covers $\mathcal{E}$. Since each f_k is unif. cont. $\exists \delta > 0$ s.t.

$$|s - t| < \delta \Rightarrow |f_k(s) - f_k(t)| < \frac{\varepsilon}{3} \text{ for each } k = 1, \dots n.$$

For some k, $f \in B(f_k, \frac{\varepsilon}{3})$. Thus

$$|s-t| < \delta \Rightarrow |f(s) - f(t)| \leq |f(s) - f_k(s)| + |f_k(s) - f_k(t)| + |f_k(t) - f(t)|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Thus, $\mathcal{E}$ is eq. cont. $\blacksquare$