Ch 4
Sec 4

Uniform Approximation in C^0

Thm Weierstrass Approximation Thm.
The set of polynomials is dense in $C^0([a,b], \mathbb{R})$.

The textbook gives two proofs. I'll cover the first and you should read/study the second on your own.

The first proof uses Bernstein polynomials. They are also used in computer graphics to create Bézier curves. We will restrict out attention to the interval $[0,1]$ for simplicity.

Def For each $n \in \mathbb{N} \cup \{0\}$ there are $n+1$ Bernstein basis polynomials of degree n. They are

$$b_{k,n} = \binom{n}{k} x^k (1-x)^{n-k} \quad \text{for } k = 0, 1, 2, \ldots, n,$$

where,

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

Next we list a few.

$$b_{00}(x) = 1$$
$$b_{01}(x) = (1-x), \quad b_{11}(x) = x$$
$$b_{02}(x) = (1-x)^2, \quad b_{12}(x) = 2x(1-x), \quad b_{22}(x) = x^2$$
$$b_{03}(x) = (1-x)^3, \quad b_{13}(x) = 3x(1-x)^2, \quad b_{23}(x) = 3x^2(1-x), \quad b_{33}(x) = x^3.$$
etc.

Then any polynomial of the form

$$\sum_{k=0}^{n} c_k b_{kn}(x),$$

is called a Bernstein poly. of degree n.

Facts We state and proof two handy facts about Bernsteien poly's.

$$\sum_{k=0}^{n} b_{k,n}(x) = 1. \qquad (\star)$$

Thus for each n the set $\{ b_{k,n} \}$ forms what is called a "partition of unity." The proof follows from the binomial thm:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}, \qquad (bt).$$

by letting $y = 1-x$.

The other identity we will need is

$$\sum_{k=0}^{n} (k - nx)^2 b_{k,n}(x) = nx(1-x). \qquad (\#)$$

The proof of this is a little harder. It goes like this. Apply $\frac{\partial}{\partial x}$ to both sides of (bt), twice. Thus,

$$n(x+y)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k x^{k-1} y^{n-k} \quad \text{and}$$

$$n(n-1)(x+y)^{n-2} = \sum_{k=0}^{\infty} \binom{n}{k} k(k-1) x^{k-2} y^{n-k}.$$

Now let $y = 1-x$ is both of these equations. Thus,

$$n = \sum_{k=0}^{n} \binom{n}{k} k x^{k-1} (1-x)^{n-k} \quad \text{and}$$

$$n(n-1) = \sum_{k=0}^{n} \binom{n}{k} k(k-1) x^{k-2} (1-x)^{n-k}.$$

Multiply the first by x and the second by x^2. Thus,

$$nx = \sum_{k=0}^{n} \binom{n}{k} k x^{k} (1-x)^{n-k} = \sum_{k=0}^{n} k\, b_{kn}(x) \quad \text{and}$$

$$n(n-1)x^2 = \sum_{k=0}^{n} \binom{n}{k} k(k-1) x^{k} (1-x)^{n-k} = \sum_{k=0}^{n} k(k-1)\, b_{kn}(x)$$

$$= \sum_{k=0}^{n} k^2 b_{nk}(x) - \underbrace{\sum_{k=0}^{n} k\, b_{kn}(x)}_{= nx}$$

Hence,

$$\sum_{k=0}^{n} k^2 b_{k,n}(x) = n(n-1)x^2 + nx.$$

Now we go back to (#):

$$\sum_{k=0}^{n} (k-nx)^2 b_{kn}(x) = \sum_{k=0}^{n} k^2 b_{kn}(x) - 2nx \sum_{k=0}^{n} k b_{kn}(x) + n^2x^2 \sum_{k=0}^{n} b_{kn}(x)$$

$$= n(n-1)x^2 + nx - 2nx(nx) + n^2x^2$$

$$= \underline{n^2x^2} \underbrace{-nx^2 + nx} - \underline{2n^2x^2} + \underline{n^2x^2}$$

$$= nx(1-x).$$

Thus (#) is proved. $\square$

Pf (Proof of the Weierstrass Approx. Thm)

Let $f \in C^0([0,1], \mathbb{R})$. For each $n \in \mathbb{N}$ define

$$p_n(x) = \sum_{k=0}^{n} \binom{n}{k} c_k x^k (1-x)^{n-k} = \sum_{k=0}^{n} c_k b_{kn}(x)$$

where $c_k = f(\frac{k}{n})$. (Of course, c_k depends on n and k.)

Note, $f(x) = f(x) \cdot 1 = f(x) \sum_{k=0}^{n} b_{kn}(x) = \sum_{k=0}^{n} f(x) b_{kn}(x)$.

We will show $p_n \rightrightarrows f$. Notice that as $n \to \infty$ the partition $\{\frac{0}{n}, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, \frac{n}{n}\}$ gets finer.

Fix a value of $x \in [0,1]$ for now. Let $\varepsilon > 0$. Since $f(x)$ is uniformly cont. on $[0,1]$, $\exists\, \delta > 0$, s.t.

$$|t-s| < \delta \Rightarrow |f(t) - f(s)| < \frac{\varepsilon}{2}.$$

Define

$$K_n = \{k \in \{0,1,2,\dots,n\} \mid |\tfrac{k}{n} - x| < \delta\} \text{ and}$$

$$K_n' = \{0,1,2,\dots,n\} - K_n.$$

Now,

$$|p_n(x) - f(x)| = \left|\sum_{k=0}^{n} (c_k - f(x)) b_{kn}(x)\right| \le \sum_{k=0}^{n} |c_k - f(x)| b_{kn}(x)$$

$$= \sum_{k \in K_n} |c_k - f(x)| b_{kn}(x) + \sum_{k \in K_n'} |c_k - f(x)| b_{kn}(x).$$

Since $c_k = f(\frac{k}{n})$, $|c_k - f(x)| < \frac{\varepsilon}{2}$ for $k \in K_n$.

Thus, $\sum_{k \in K_n} |c_k - f(x)| b_{kn}(x) < \frac{\varepsilon}{2} \sum_{k \in K_n} b_{kn}(x) \leq \frac{\varepsilon}{2}$

For $k \in K_n'$ we have

$$\left|\frac{k}{n} - x\right| \geq \delta \Rightarrow |k - nx|^2 \geq (n\delta)^2$$

Now by (#) we have

$$nx(1-x) = \sum_{k=0}^{n} (k-nx)^2 b_{kn}(x) \geq \sum_{k \in K_n'} (k-nx)^2 b_{kn}(x)$$

$$\geq \sum_{k \in K_n'} (n\delta)^2 b_{kn}(x).$$

Therefore,

$$\sum_{k \in K_n'} b_{kn}(x) \leq \frac{nx(1-x)}{(n\delta)^2} \leq \frac{1}{4n\delta^2},$$

Since the max of $x(1-x)$ over $[0,1]$ is $\frac{1}{4}$.

Let $M = \|f\|$. Then $\|c_k - f(x)\| = \|f(\frac{k}{n}) - f(x)\| \leq 2M$.

Thus,

$$\sum_{K_n'} |c_k - f(x)| b_{kn}(x) \leq \frac{M}{2n\delta^2} < \frac{\varepsilon}{2},$$

for n large enough.

Thus, $\exists N$ s.t. $n \geq N \Rightarrow |p_n(x) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

But for uniform convergence N must be independent of our choice of $x \in [0,1]$. The sets K_n, K_n' clearly did depende on x. But, the bounds on the two sums, and , did not. Thus the convergence is uniform.

Ex Let $f: [0,1] \to \mathbb{R}$ be given by $f(x) = |x - \frac{1}{2}|$. Find the smallest n so that $\|p_n - f\| < 0.1$.

On the next page is a Maple worksheet. Letting $n = 15$ gives $\|p_n - f\| = 0.1047\ldots$, while $n = 16$ gives $\|p_n - f\| = 0.09819\ldots$. Thus, the smallest value for n is 16.

Repeat for $f(x) = ||x - \frac{1}{2}| - \frac{1}{4}|$ and $\|p_n - f\| < 0.02$. This time we get $n = 398$.

Bernstein Polynomial Approximations

Find the first value of n so the the nth Bernstein polynomial for f(x) = |x−0.5| has $||p_n - f|| < 0.1$.

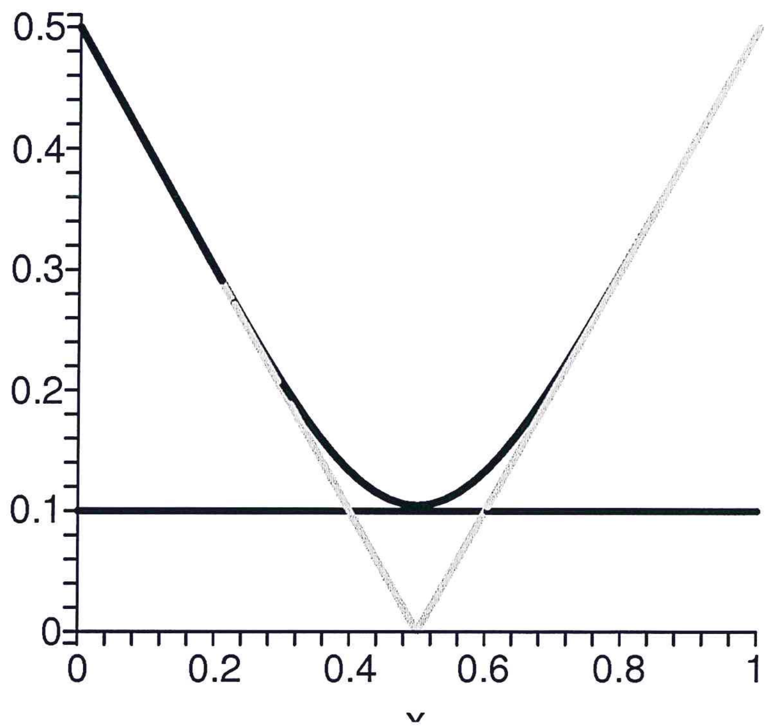
```
> pn:= x-> sum((n!/k!/(n-k)!)*abs(k/n-0.5)*x^k*(1-x)^(n-k),k=0..n);
```

$$pn := x \rightarrow \sum_{k=0}^{n} \frac{n! \left| \frac{k}{n} - 0.5 \right| x^k (1-x)^{(n-k)}}{k!\,(n-k)!}$$

```
> n:=15;
```

$$n := 15$$

```
> plot([0.1,abs(x-0.5),pn(x)],x=0..1,color=[black,gray,black],
  linestyle=[2,1,1],thickness=2);
```




```
> pn(0.5);
```

$$0.1047363281$$

```
> n:=16;
```

$$n := 16$$

```
> pn(0.5);
```

$$0.09819030762$$

The answer is n=16.

Repeat for f(x) = | |x−0.5| − 0.25| and $||p_n - f||<0.02$.

```
> f:=x-> abs(abs(x-0.5)-0.25);
```

$$f := x \to \left| \, |x - 0.5| - 0.25 \right|$$

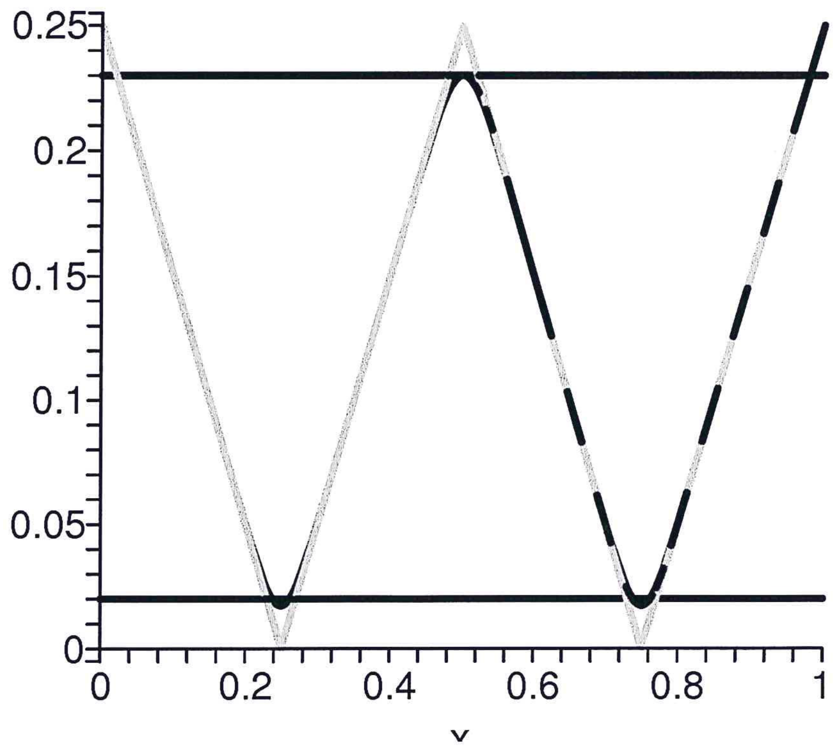
```
> pn:= x-> sum((n!/k!/(n-k)!)*f(k/n)*x^k*(1-x)^(n-k),k=0..n);
```

$$pn := x \to \sum_{k=0}^{n} \frac{n!\, f\left(\frac{k}{n}\right) x^k (1-x)^{(n-k)}}{k!\,(n-k)!}$$

```
> n:=398;
```

$$n := 398$$

```
> plot([0.02,0.23,f(x),pn(x)],x=0..1,color=[black,black,gray,black],
  linestyle=[2,2,1,1],thickness=2);
```




```
> pn(0.5);
```

$$0.2300153875$$

The answer is n=398.

The Stone-Weierstrass Thm.

Defs Let M be a compact metric space. (It follows that M is complete.) Let $\mathcal{A} \subset C^0(M, \mathbb{R})$. Then $\mathcal{A}$ is called a function algebra if $\forall f, g \in \mathbb{R}, c \in \mathbb{R}$ we have $f+g$, cf, $f \cdot g$ are in $\mathcal{A}$.

A function algebra vanishes at $p \in M$ if $\forall f \in \mathcal{A}$, $f(p) = 0$.

A func. alg. separates points if $\forall p_1, p_2 \in M$, $p_1 \neq p_2$ $\exists f \in \mathcal{A}$ s.t. $f(p_1) \neq f(p_2)$.

Ex Polynomials in $C^0([a,b], \mathbb{R})$ form a nonvanishing func. alg that sep. pts.

Ex $\{ p(x)(x-3) \mid p(x) \text{ any poly} \}$ vanishes at $p=3$.

Ex $\{ p(x)(x^2-1)+2 \mid p(x) \text{ any poly} \}$ does not sep. 1 and -1.

Thm (Stone-Weierstrass) Let M be a compact m. sp. and $\mathcal{A} \subset C^0(M, \mathbb{R})$ a func. alg. that vanishes nowhere and sep. pts. Then $\mathcal{A}$ is dense in $C^0(M, \mathbb{R})$.

This is a generalization of the W. Approx Thm. We need some lemmas.

Lemma If $\mathcal{A}$ vanishes nowhere and sep. pts, then $\forall\, p_1, p_2 \in M$, $p_1 \neq p_2$ and $c_1, c_2 \in \mathbb{R}$, $\exists\, f \in \mathcal{A}$ s.t.

$$f(p_1) = c_1 \text{ and } f(p_2) = c_2.$$

pf $\exists\, g_1, g_2 \in \mathcal{A}$ s.t. $g_1(p_1) \neq 0$ and $g_2(p_2) \neq 0$.

Let $g(x) = [g_1(x)]^2 + [g_2(x)]^2$. Then $g \in \mathcal{A}$ and $g(p_1) \neq 0$, $g(p_2) \neq 0$. $\exists\, h \in \mathcal{A}$ s.t. $h(p_1) \neq h(p_2)$.

Consider the matrix $H = \begin{bmatrix} g(p_1) & g(p_1)h(p_1) \\ g(p_2) & g(p_2)h(p_2) \end{bmatrix} = \begin{bmatrix} a & ab \\ c & cd \end{bmatrix}$.

Now $a \neq 0$, $c \neq 0$ and $b \neq d$. Thus $\det H = acd - abc = ac(b-d) \neq 0$. Therefore, $\exists!$ solution to

$$\begin{bmatrix} a & ab \\ c & cd \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} c_1 \\ c_1 \end{bmatrix}.$$

Let $f(x) = p\,g(x) + q\,g(x)h(x)$. Then $f \in \mathcal{A}$. We compute

$$f(p_1) = pa + qab = c_1 \quad \text{and}$$

$$f(p_2) = pc + qcd = c_2.$$

Lemma Let $\mathcal{A}$ be a func. alg. Then $\bar{\mathcal{A}} \subset C^0(M, \mathbb{R})$ and is a func. alg

Pf C^0 is closed, so $\mathcal{A} \subset C^0 \Rightarrow \bar{\mathcal{A}} \subset C^0$.

Let $f, g \in \bar{\mathcal{A}}$ and $c \in \mathbb{R}$. $\exists$ seq's in $\mathcal{A}$ s.t. $f_n \to f$ and $g_n \to g$. Thus,

$$(f_n + g_n) \to f+g \Rightarrow f+g \in \bar{\mathcal{A}},$$
$$cf_n \to cf \Rightarrow cf \in \bar{\mathcal{A}},$$
$$f_n \cdot g_n \to f \cdot g \Rightarrow fg \in \bar{\mathcal{A}}.$$

Thus $\bar{\mathcal{A}}$ is a func. alg. $\blacksquare$

Lemma $f \in \bar{\mathcal{A}} \Rightarrow |f| \in \bar{\mathcal{A}}$.

Note $f \in \mathcal{A} \not\Rightarrow |f| \in \mathcal{A}$. Let $\mathcal{A}$ = all polynomials. $|x^2 - 1| \notin \mathcal{A}$.

Pf Let $f \in \bar{\mathcal{A}}$ and $M = \|f\|$. Let $\varepsilon > 0$. We apply the Weierstrass Approx. Thm to $|x| \in C^0([-M, M], \mathbb{R})$ to get a poly $p(x)$ s.t.

$$\sup\{|p(x) - |x|| : x \in [-M, M]\} < \frac{\varepsilon}{2}.$$

It follows that the constant term of the poly $p(x)$ has $|p(0)| < \varepsilon/2$. Let $q(x) = p(x) - p(0)$.

Then $\sup\{|q(x)-|x||: x\in[-M,M]\}<\varepsilon$. Let $a_1,\dots a_n$ be the coeff's. of $q(x)$:

$$q(x)=a_1x+a_2x^2+\cdots+a_nx^n.$$

Now define $g(x)=q(f(x))=a_1f(x)+a_2f^2(x)+\cdots+a_nf^n(x)$.

By the previous lemma $g\in\bar{\mathcal{A}}$. Write $y=f(x)$. Then $y\in[-M,M]$, $\forall x\in M$. Thus

$$|g(x)-|f(x)||=|q(y)-|y||<\varepsilon.$$

Since this can be done for any $\varepsilon>0$ and $\bar{\mathcal{A}}$ is closed, we have $|f|\in\bar{\mathcal{A}}$. $\triangle$

Lemma Let $f,g\in\bar{\mathcal{A}}$. Then $\max(f,g)$ and $\min(f,g)$ are in $\bar{\mathcal{A}}$.

Pf

$$\max(f,g)=\frac{f+g}{2}+\frac{|f-g|}{2}\in\bar{\mathcal{A}}$$

$$\min(f,g)=\frac{f+g}{2}-\frac{|f-g|}{2}\in\bar{\mathcal{A}} \quad \blacksquare$$

Note By induction, this extends to max and min of $f_1,f_2,\dots,f_n$.

Pf of the S-W Thm

Let M be compact, $\mathcal{a} \subset C^0(M, \mathbb{R})$ a func. alg. Let $F \in C^0(M, \mathbb{R})$ and let $\varepsilon > 0$. We claim $\exists\ G \in \bar{\mathcal{a}}$ s.t.

$$F(x) - \varepsilon < G(x) < F(x) + \varepsilon, \quad x \in M.$$

Since this will hold $\forall\ \varepsilon > 0$, we get $F \in \bar{\bar{\mathcal{a}}} = \bar{\mathcal{a}}$. Hence, $\bar{\mathcal{a}} = C^0(M, \mathbb{R})$. It will take a bit work to get there, since, after all, we have to use all those lemmas. Hang on to your seats!

Pick any two distinct pts $p, q \in M$. $\exists\ H \in \mathcal{a}$ s.t. $H(p) = F(p)$ and $H(q) = F(q)$. (First lemma.)

At $x = q$ we have $\varepsilon + H(x) - F(x) = \varepsilon > 0$. By continuity $\exists$ nbhd U_q of q s.t. for $x \in U_q$, $\varepsilon + H(x) - F(x) > 0$, or equivalently $F(x) - \varepsilon < H(x)$.

Let q vary over M, and for each select an $H_q \in \mathcal{a}$ and a nbhd U_q as above: $\forall x \in U_q$, $F(x) - \varepsilon < H_q(x)$. ($p$ is held fixed and nothing goes "wrong" if $q = p$.) The collection $\{U_q\}_{q \in M}$ is an open cover of M, and M is compact, so we can pass to a finite subcover, say $\{U_{q_1}, U_{q_2}, \ldots, U_{q_n}\}$.

Now define $G_p(x) = \max\{H_{q_1}(x), H_{q_2}(x), \ldots, H_{q_n}(x)\}$.

Then $G_p \in \bar{\mathcal{A}}$, $G_p(p) = F(p)$ and $F(x) - \varepsilon < G_p(x)$, $\forall x \in M$.

At $x = p$, $\varepsilon + F(x) - G_p(x) = \varepsilon > 0$. By continuity $\exists$ nbhd V_p of p s.t. $\forall x \in V_p$, $\varepsilon + F(x) - G_p(x) > 0$. Thus,

$$G_p(x) < F(x) + \varepsilon, \quad \forall x \in V_p.$$

Let p vary over M. For each there is a G_p and V_p as above. The collection $\{V_p\}_{p \in M}$ is an open cover. Let $\{V_{p_1}, V_{p_2}, \ldots V_{p_m}\}$ be a finite subcover. For $i = 1, 2, \ldots m$ we have

$$G_{p_i}(x) < F(x) + \varepsilon, \quad \forall x \in V_{p_i} \text{ and}$$

$$F(x) - \varepsilon < G_{p_i}(x), \quad \forall x \in M.$$

Let $G(x) = \min\{G_{p_1}(x), G_{p_2}(x), \ldots, G_{p_m}(x)\}$.

Then $G \in \bar{\mathcal{A}}$ and

$$F(x) - \varepsilon < G(x) < F(x) + \varepsilon, \quad \forall x \in M.$$

$\boxed{\text{Done!}}$

Application   Fourier Series

The set of functions from $\mathbb{R} \to \mathbb{R}$ that are periodic with the same period, say 2π, can be regarded as functions from $S^1 \to \mathbb{R}$. Consider $C^0(S^1, \mathbb{R})$.

Let $\mathcal{J} \subset C^0(S^1, \mathbb{R})$ be the vector space spanned by

$$\{1, \cos(x), \cos(2x), \cos(3x), \dots, \sin(x), \sin(2x), \dots\}.$$

With the aid of some trig identities you can show $\mathcal{J}$ is closed under multiplication. For example

$$\sin(mx)\sin(nx) = 2\cos((m-n)x) - 2\cos((m+n)x).$$

Therefore $\mathcal{J}$ is a func. alg. Since S^1 is compact the Stone-Weierstrass Thm says*

$$\overline{\mathcal{J}} = C^0(S^1, \mathbb{R}).$$

This can be extended functionspaces that are periodic in n variables. Now the domain is the n-dimensional torus,

$$T^n = S^1 \times S^1 \times \cdots \times S^1 \text{ (n times)},$$

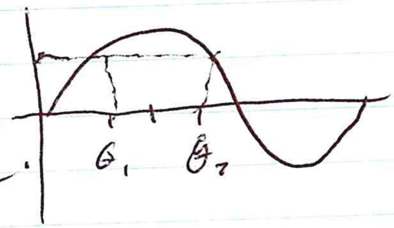
which is compact.

* Also need nonvanishing and separates points.
(next page)

Since $1 \in \tilde{J}$ it is nonvanishing.

Let $\theta_1, \theta_2 \in S^1$, $\theta_1 \neq \theta_2$. Then $\sin\theta_1 \neq \sin\theta_2$ unless they are equal distance from $\frac{\pi}{2}$ or $\frac{3\pi}{2}$

If this is the case use $\cos\theta$, $\cos(\theta_1) \neq \cos\theta_2$.



<u>Note:</u> We are dealing with uniform convergence. Fourier Series are often used when only point-wise convergence is needed, For example, the Fourier Series for can converge to discontinuous functions like a square wave.