Ch 4 Sec 5

Contractions and Diff Eqs.

Def Let $f: M \to M$. If $p \in M$ and $f(p) = p$, then p is called a fixed point of f.

Def Let $f: M \to M$. If $\exists\, k \in [0,1)$ s.t. $\forall x, y \in M$,

$$d(f(x), f(y)) \le k\, d(x,y),$$

then f is called a contration.

Thm [Banach Contraction Principle or the Contraction Mapping Theorem] Let M be a complete metric space. Let $f: M \to M$ be a contraction. The f has a unique fixed point.

Pf Uniqueness is easy, so we'll do that first. Suppose $p, q \in M$ are distinct fixed points. Then $d(p,q) \neq 0$. But

$$d(f(p), f(q)) = d(p,q) \quad \text{and}$$

$$d(f(p), f(q)) \le k\, d(p,q) < d(p,q).$$

Thus $d(p,q) < d(p,q)$. Contradiction.

Choose any $x_0 \in M$. Let $x_n = f^n(x_0)$ $(f^n = f \circ f \circ \cdots \circ f)$.
Then for any $n \in \mathbb{N}$ we have

$$d(x_n, x_{n+1}) \leq k\, d(x_{n-1}, x_n) \leq k^2 d(x_{n-2}, x_{n-1}) \leq \cdots \leq k^n d(x_0, x_1).$$

We use this idea to show (x_n) is Cauchy.
Suppose $m < n$. Then

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ &\leq k^m d(x_0, x_1) + k^{m+1} d(x_0, x_1) + \cdots + k^{n-1} d(x_0, x_1) \\ &= k^m (1 + k + \cdots + k^{n-m-1})\, d(x_0, x_1) \\ &\leq k^m \left(\sum_{i=0}^{\infty} k^i \right) d(x_0, x_1) = \frac{k^m}{1-k} d(x_0, x_1). \end{aligned}$$

Let $\varepsilon > 0$. Since $k \in [0,1)$, $\exists\, N$ s.t. $m \geq N$ implies

$$\frac{k^m}{1-k} d(x_0, x_1) < \varepsilon.$$

Thus (x_n) is Cauchy. Since M is complete $\exists\, p \in M$ s.t. $x_n \to p$. This is our fixed point.

$$\begin{aligned} d(p, f(p)) &\leq d(p, x_n) + d(x_n, f(x_n)) + d(f(x_n), f(p)) \\ &\leq d(p, x_n) + k^n d(x_0, x_1) + k\, d(x_n, p) \\ &\longrightarrow 0 \text{ as } n \to \infty. \end{aligned}$$

Thus, $d(p, f(p)) = 0$ and so $f(p) = p$. 

Thm (Brouwer's Fixed Point Thm) Suppose $f: B^n \to B^n$ is continuous where B^n is the closed unit ball in $\mathbb{R}^n$. Then f has a fixed pt (not necessarily unique)

pf The textbook does the $n=1$ case. For the rest, take Math 430.

Cor Let X be a metric space that is homeomorphic to B^n. Let $f: X \to X$, be cont. Then f has a fixed pt.

pf Let $h: X \to B^n$ be a homeo. Consider the diagram.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & X \arrow[d, "h"] \\ B^n \arrow[r, dashed, "g"'] & B^n \end{tikzcd}$$

where g is defined by $g = hfh^{-1}$. Then g has a fixed pt $x \in B^n$. Let $y = h^{-1}(x)$. Compute

$$f(y) = f(h^{-1}(x)) = f(h^{-1}(g(x)))$$

$$h(f(y)) = h(f(h^{-1}(g(x)))) = g(g(x)) = x$$

$$f(y) = h^{-1}(x) = y.$$

∎

ODEs: Picard's Existence Thm

Def Let $F: \mathbb{R}^m \to \mathbb{R}^m$ be regarded as a vector field. A solution curve is a curve $\gamma: \mathbb{R} \to \mathbb{R}^m$ s.t. (or (a,b))

$$\gamma'(t) = F(\gamma(t)). \qquad (\star)$$

If $\gamma(0) = p$, we say γ is a solution to F with initial condition $\gamma(0) = p$.

Rmk This can be generalized to vector fields on manifolds, $F: M \to \mathbb{R}^m$ (M has dim m) and $\gamma: \mathbb{R} \to M$. But, the idea of a derivative has to be generalized. This is done in courses on differential topology and diff. geom. See *Calculus on Manifolds* by Spivak.

Def A v.f. $F: \mathbb{R}^m \to \mathbb{R}^m$ satisfies a Lipschitz condition if $\exists\, L > 0$ s.t.

$$|F(x) - F(y)| \leq L|x - y|.$$

Note Lip. $\Rightarrow$ continuity, but not much more. Any differientialbe v.f. with bdd derivative is Lip.

Note The eq. $(\star)$ + an init. cond. can be rewritten as

$$\gamma(t) = p + \int_0^t F(\gamma(s))\, ds,$$

where integral is defined to be the vector of the integrals of each component. See text for details.

Thm (Picard's Thm) Let F be a Lip. v.f. defined on an open set $U \subset \mathbb{R}^m$ and let $p \in U$. Then $\exists$ $a < 0 < b$ and $\gamma : (a,b) \to U$ that solves

$$\gamma'(t) = F(\gamma(t)) \text{ and } \gamma(0) = p,$$

and is unique on (a,b).

Pf Since F is Lip it is cont. Thus, on any compact nbhd of p there is a constant M s.t. $|F(x)| \leq M$ for all x in this compact nbhd. We will use $\overline{B(p,r)}$, the closed ball of radius r centered at p for this nbhd. Let L be a Lip const. for F over U. Choose $J > 0$ s.t.,

$$JM \leq r \quad \text{and} \quad JL < 1.$$

Let $C = \{$ cont. functions $\gamma : [-J, J] \to \overline{B(p,r)} \}$. Give C the sup metric

$$d(\gamma_1, \gamma_2) = \sup\{|\gamma_1(t) - \gamma_2(t)| : t \in [-J, J]\}.$$

Then C is a complete metric space. Why? $C = C^0([-J,J], \overline{B(p,r)})$. We know $C^0([-J,J], \mathbb{R})$ is a complete space. Generalize. Explain.

For $\gamma \in C$ define

$$\Phi(\gamma) = p + \int_0^t F(\gamma(s))\, ds.$$

It is a function of t.

Then $\Phi : \mathcal{C} \to \mathcal{C}$. We will show that Φ is a contraction and that its "fixed curve" is a solution curve.

First, we check that $\Phi(\gamma) \in \mathcal{C}$. For $t \in [-T, T]$

$$\left| \Phi(\gamma)(t) - p \right| = \left| \int_0^t F(\gamma(s))\, ds \right| \leq TM \leq r.$$

Thus $\Phi(\gamma)([-T, T]) \subset \overline{B(p, r)}$. Thus $\Phi(\gamma) \in \mathcal{C}$.

Now Φ is a contraction since,

$$d(\Phi(\gamma_1), \Phi(\gamma_2)) = \sup_{t \in [-T,T]} \left\{ \left| \int_0^t F(\gamma_1(s)) - F(\gamma_2(s))\, ds \right| \right.$$

$$\leq T \sup_{s \in [-T,T]} \left\{ \left| F(\gamma_1(s)) - F(\gamma_2(s)) \right| \right\}$$

$$\leq T \sup_{s \in [-T,T]} \left\{ L \left| \gamma_1(s) - \gamma_2(s) \right| \right\} \leq TL\, d(\gamma_1, \gamma_2).$$

But since $TL < 1$ we have that Φ is a contraction.

Let γ be the unique fixed curve/point. Then

$$\gamma(t) = \Phi(\gamma)(t) = p + \int_0^t F(\gamma(s))\, ds.$$

By definition any solution is a fixed curve/point of Φ. Thus our solution is unique. $\square$

Notice this proof gives a method for finding the solution: we make a guess, and then apply Φ over and over.

Ex Let $F(x) = \frac{1}{2}x$, $x \in \mathbb{R}$, be a vector field on $\mathbb{R}$. Find $\gamma(t)$ s.t. $\gamma'(t) = F(\gamma(t)) = \frac{1}{2}\gamma(t)$ and $\gamma(0) = 1$.

[We are just solving $y' = \frac{1}{2}y$, $y(0) = 1$. You should know the answer is $e^{t/2}$.]

$\boxed{\text{Sol}}$ F is Lip with $L = \frac{1}{2}$. For our guess, let $\gamma_0(t) = 1$.

$$\gamma_1(t) = 1 + \int_0^t \frac{1}{2}\cdot 1 \, ds = 1 + \frac{t}{2}.$$

$$\gamma_2(t) = 1 + \int_0^t F\left(1 + \frac{s}{2}\right) ds = 1 + \frac{1}{2}\int_0^t 1 + \frac{s}{2} \, ds = 1 + \frac{t}{2} + \frac{t^2}{4\cdot 2}$$

$$\gamma_3(t) = 1 + \frac{1}{2}\int_0^t 1 + \frac{s}{2} + \frac{s^2}{4\cdot 2} \, ds = 1 + \frac{t}{2} + \frac{t^2}{4\cdot 2} + \frac{t^3}{8\cdot 2\cdot 3}$$

$$\gamma_4(t) = 1 + \frac{1}{2}\int_0^t 1 + \frac{s}{2} + \frac{s^2}{4\cdot 2} + \frac{s^3}{8\cdot 2\cdot 3} \, ds = 1 + \frac{t}{2} + \frac{t^2}{4\cdot 2} + \frac{t^3}{8\cdot 2\cdot 3} + \frac{t^4}{16\cdot 4!}$$

$$\vdots$$

$$\gamma_n(t) = 1 + \frac{t}{2} + \frac{1}{2!}\left(\frac{t}{2}\right)^2 + \frac{1}{3!}\left(\frac{t}{2}\right)^3 + \frac{1}{4!}\left(\frac{t}{2}\right)^4 + \cdots + \frac{1}{n!}\left(\frac{t}{2}\right)^n$$

Hence $\gamma_n(t) \to e^{t/2}$.

A standard textbook for diff eq's is "Elementary Differential Equations and Boundary Value Problems" by Boyce and DiPrima. The existence and uniqueness theorem, Thm 2.8.1, in the 10th edition, is stated this way:

Consider the initial value problem

$$y' = f(t, y), \quad y(0) = 0. \quad (\star)$$

The goal is to find $y(t)$. If f and $\partial f/\partial y$ are continuous in a rectangle $R: |t| \le a, |y| \le b$, then there is some interval $|t| \le h \le a$ in which $(\star)$ has a unique solution.

Study their proof and compare it to Picard's Thm.