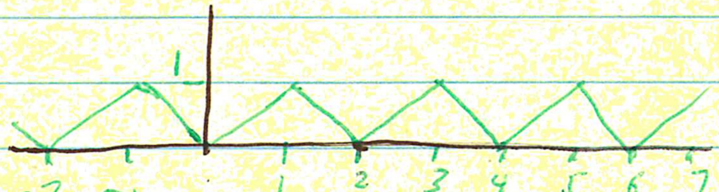
Ch 4

Section 7: ~~Continuous~~ Continuous Nowhere Differentiable Functions are Everywhere

First we shall construct an example of a continuous function $f: \mathbb{R} \to \mathbb{R}$ s.t. $f'(x)$ does not exist $\forall x \in \mathbb{R}$. Then we will prove Baire's Thm and then use it to show that among continuous functions in $C^0([0,1], \mathbb{R})$, the nowhere differentiable are the "most common."

Thm $\exists$ a continuous $f: \mathbb{R} \to \mathbb{R}$ that is nowhere differentiable.

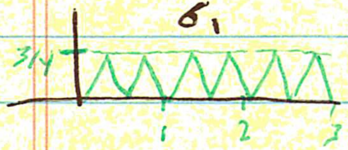
Pf Let $\sigma_0(x) = \begin{cases} x - 2n & \text{if } 2n \le x \le 2n+1 \\ (2n+2) - x & \text{if } 2n+1 \le x \le 2n+2 \end{cases}$

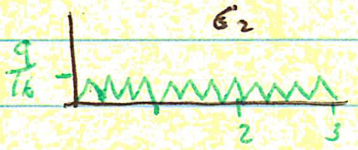


It is cont.

Let $\sigma_k(x) = \left(\frac{3}{4}\right)^k \sigma_0(4^k x)$.

The period of σ_k is $T_k = \frac{2}{4^k}$



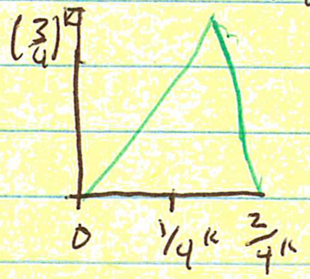


According to the Weierstrass M-test (pg 217), since $\|\sigma_k\| = (3/4)^k$ and $\sum_{k=0}^{\infty} (3/4)^k$ converges, the sum $\sum_{k=0}^{\infty} \sigma_k(x)$ converges uniformly to a continuous function.

Let

$$f(x) = \sum_{k=0}^{\infty} \sigma_k(x).$$

We claim $f'(x)$ d.n.e $\forall x \in \mathbb{R}$. The proof basically rests on the following fact: The slopes of the linear segments of σ_k are $\pm 3^k$.



slopes $= \pm 3^k$

Let $x \in \mathbb{R}$ and $\delta > 0$. Let $\delta_n = \frac{1}{2 \cdot 4^n}$ with n big enough that $\delta_n < \delta$. We define a slope function

$$S(\alpha) = \frac{f(x+\alpha) - f(x)}{\alpha},$$

for $\alpha \in (-\delta, 0) \cup (0, \delta)$. We will show that for $m \geq n$ either $|S(\delta_m)|$ or $|S(-\delta_m)|$ is $\geq \frac{1}{2}(3^m + 1)$. Thus the limit $\lim_{\alpha \to 0} S(\alpha)$ does not exist. Hence $f'(x)$ d.n.e.

Set $\alpha = \pm \delta_m$. Then $|\alpha| = \delta_m = \frac{1}{2\,4^m} =$

$$4^{k-(m+1)}\left(\frac{2}{4^k}\right) = 4^{k-(m+1)}\,\pi_k.$$

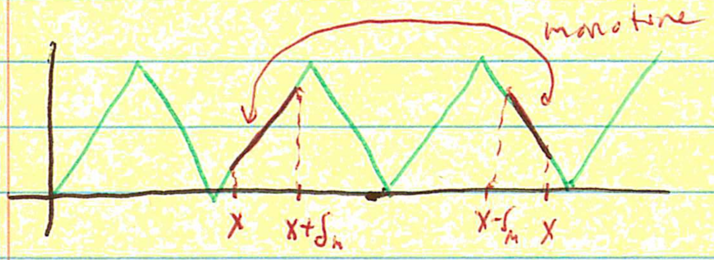
Thus, once $k > m$ we have $\sigma_k(x+\alpha) = \sigma_k(x)$. Hence,

$$S(\alpha) = \sum_{k=0}^{\infty} \frac{\sigma_k(x+\alpha) - \sigma_k(x)}{\alpha} = \sum_{k=0}^{m} \frac{\sigma_k(x+\alpha) - \sigma_k(x)}{\alpha}.$$

Consider the last nonzero term, $k=m$, $\dfrac{\sigma_m(x\pm\delta_m)-\sigma_m(x)}{\pm\delta_m}$.

On either $[x-\delta_m, x]$ or $[x, x+\delta_m]$ we see that σ_m is monotone. If the first does we use $\alpha=-\delta_m$, if the second does, we ~~have~~ use $\alpha=\delta_m$. Thus,

$$\left|\frac{\sigma_m(x+\alpha)-\sigma_m(x)}{\alpha}\right| = 3^m$$



For $0\le k<m$ we have $\left|\dfrac{\sigma_k(x\pm\delta_m)-\sigma_k(x)}{\pm\delta_m}\right|\le 3^k$

Thus,

$$|S(\alpha)| \ge 3^m-(3^{m-1}+3^{m-2}+\dots+3+1)$$

$$=3^m-\left(\frac{3^m-1}{3-1}\right)=\frac{1}{2}(3^m+1).$$

Therefore, $f'(x)$ does not exist.

Baire Spaces

Def Let X be a topological space. For each $n \in \mathbb{N}$ let $G_n \subset X$ be open and dense. Let $G = \bigcap G_n$. Sets arising in this way are sometimes called residual sets. We are interested in conditions on X that will guarantee that residual sets are themselves dense. Space with this property are called Baire spaces.

Ex Let $\{q_1, q_2, \dots\}$ be an enumeration of the rational numbers. Let $G_n = \mathbb{R} - \{q_n\}$. Then each G_n is open and dense in $\mathbb{R}$. Now $G = \bigcap G_n$ is the set of irrational numbers. As you know $\overline{G} = \mathbb{R}$.

Def Suppose each point of X either has a certain property or does not. This property is said to be generic if every point in a residual set has it.

Ex Being irrational is a generic property in $\mathbb{R}$.

We will show (later) that being nowhere differentiable is a generic property in $C^0([0,1], \mathbb{R})$. But, first we need to prove Baire's Thm.

Thm (Baire's Thm) Let M be a complete metric space. Let $\{G_k\}_{k=1}^{\infty}$ be a countable collection of open dense subsets of M. Then $G = \bigcap_{k=1}^{\infty} G_k$ is dense in M.

pf (From Royden's Real Analysis, 3rd Ed, pg 158.)

Let U be an arbitrary open subset of M. We will show that $U \cap G \neq \phi$. It follows that $\overline{G} = M$. This will be done by constructing a seq (x_n) whose limit must be in $U \cap G$.

Let $x_1 \in G_1 \cap U$, which is nonempty since G is dense. Since $G_1 \cap U$ is open $\exists$ a open ball $B_1 = B(x_1, r_1) \subset G_1 \cap U$.
Not needed here. ~~We can choose $r_1 > 0$ s.t. $\overline{B_1} \subset G_1 \cap U$.~~

Let $x_2 \in G_2 \cap B_1$. Let $B_2 = B(x_2, r_2)$ be an open ball s.t. $B_2 \subset G_2 \cap B_1$ and $r_2 < r_1/2$. Thus $\overline{B_2} \subset B_1$.

We continue inductively, forming sequences $\{x_n\}$, $\{B_n\}$ s.t.

$$B_n \subset G_n \quad \overline{B_n} \subset B_{n-1} \text{ and } r_n \to 0.$$

Then (x_n) is Cauchy since $m, n \geq N \Rightarrow x_m, x_n \in B_N$ $\Rightarrow d(x_m, x_n) < 2r_N$ and $r_N \to 0$.

Since M is complete $\exists\, x \in M$ s.t. $x_n \to x$. Next we show $x \in G$.

For $n > N$, $x_n \in B_{N+1}$. Thus, $x \in \overline{B_{N+1}} \subset B_N \subset G_N$. This holds for a $N \in \mathbb{N}$. Thus $x \in G_N$, $\forall N$ and hence $x \in G$ as claimed.

Note added: Since x is in $\overline{B_2}$ is in U, it is clear x is in U as well as G.

Rmk Baire's Thm hold if "complete metric space" is replaced with "compact Hausdorff space" See Munkres' Topology, Section 48.

Rmk The definition of a Baire Space is equivalent to the following: Given any countable collection $\{A_n\}$ of closed sets in ~~X~~ with empty interiors, their union $\cup A_n$ also has empty interior. See Munkres, Sec 48.

Other Terminology Residual subset are also called thick subsets. The complement of a residual subset is called a Meager subset or also a thin subset. See Pugh's textbook, pg 256.

A subset of a top. sp. is said to be of the first category if it was contained in the union of a countable collection of closed sets with empty interiors. Otherwise, it ~~was~~ is of the second category. In this terminology Baire's Thm is called Baire's Category Thm: In a Baire Sp no nonempty open set is of the first category. See Munkres.

Thm In $C^0([a,b],\mathbb{R})$ nowhere differentiable functions are generic.

Pf For simplicity I'll do $C^0([0,1],\mathbb{R})$. This proof is based on Glen Bredon's book Topology and Geometry, see Cor. 67.6, pages 60-61.

Let $U_n = \{ f \in C^0 \,/\, \forall t \in [0,1], \exists\, s \in [0,1]-\{t\}$ s.t.

$$\left| \frac{f(t)-f(s)}{t-s} \right| > n \}.$$

If $f(x) = 2nx$, then $f \in U_n$. Hence $U_n \neq \emptyset$.

Our proof is done in the following steps.

Claim I: If $f \in \bigcap U_n$ then f is nowhere diff.

Claim II: Each U_n is dense.

Claim III: Each U_n is open.

The result then follows by Baire's Thm.

Claim II Let $f \in \bigcap U_n$. Suppose $f'(t)$ exists for some $t \in [0,1]$.

Consider $\left|\frac{f(t)-f(s)}{t-s}\right|$ as a function of s. $\exists\, \delta > 0$ s.t.

$s \in ((t-\delta, t) \cup (t, t+\delta)) \cap [0,1] \Rightarrow$

$$\left|\frac{f(t)-f(s)}{t-s}\right| < |f'(t)|+1.$$

On the compact set $[0,1]-(t-\delta, t+\delta)$

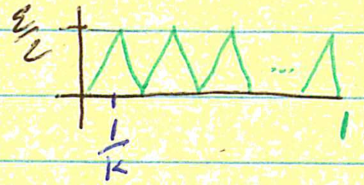
$\left|\frac{f(t)-f(s)}{t-s}\right|$, as a function of s, is cont. and hence bdd,

by say M. Let $n > \max\{M, |f'(t)|+1\}$.

Then $f \notin U_n$. This is a contradiction.

Hence $f'(t)$ does not exist for $t \in [0,1]$.

Claim II We will show each U_n is dense. Let $f \in C^0$ and $\varepsilon > 0$. We will construct a $g \in U_n$ s.t. $\|f-g\| < \varepsilon$. If f was the zero function we know how to do this. Let $g(x)$ be...



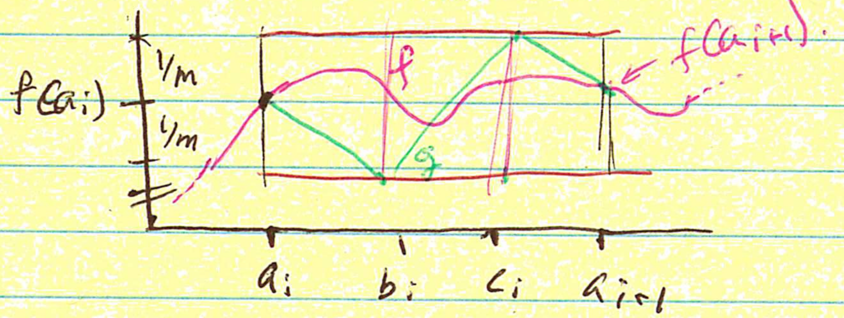
where $\frac{\varepsilon k}{2} > n$.

In general pick m s.t. $\frac{2}{m} < \varepsilon$. By uniform cont. of f $\exists\, k$ s.t.

$$|x-y| < \frac{1}{k} \Rightarrow |f(x) - f(y)| < \frac{1}{m}.$$

Choose k even larger, if needed, so that $k > mn$.

Let $a_i = \frac{i}{k}$, $i = 0, \dots, k$. This gives a partition of $[0,1]$. Subdivide it further using $b_i = a_i + \frac{1}{3k}$ and $c_i = a_i + \frac{2}{3k}$, $i = 0, \dots, k-1$. Define $g(x)$ on each $[a_i, a_{i+1}]$ like below.



This gives $g : [0,1] \to \mathbb{R}$ that is cont. and

$$\|f - g\| \le \frac{2}{m} < \varepsilon.$$

We examine the three segments that make up g on $[a_i, a_{i+1}]$. On $[a_i, b_i]$ and $[b_i, c_i]$ the $|\text{slope}|$ of g is greater than n (as we shall see). On $[c_i, a_{i+1}]$ it may not be, but we can get around this. The goal is to find for each t an $s \neq t$ s.t. $\left|\frac{g(t)-g(s)}{t-s}\right| > n$. Hence $g \in U_n$.

Suppose $t \in [a_i, b_i]$. Choose any $s \in [a_i, b_i] - \{t\}$. Then

$$\left|\frac{g(t)-g(s)}{t-s}\right| = |\text{slope}| = \left|\frac{-1/m}{1/3k}\right| = \frac{3k}{m} > \frac{3nm}{m} = 3n > n.$$

Suppose $t \in [b_i, c_i]$. Choose $s \in [b_i, c_i] - \{t\}$. Then

$$\left|\frac{g(t)-g(s)}{t-s}\right| = |\text{slope}| = \left|\frac{2/m}{1/3k}\right| = \frac{6k}{m} > \frac{6mn}{m} = 6n > n.$$

For $t \in [c_i, a_{i+1}]$ we have two cases.

(i) Suppose $f(a_{i+1}) \leq f(a_i)$. Choose $s \in [c_i, a_{i+1}] - \{t\}$. Then $$\left|\frac{g(t)-g(s)}{t-s}\right| = |\text{slope}| \geq \left|\frac{1/m}{1/3k}\right| = \frac{3k}{m} > n$$

(ii) Suppose $f(a_{i+1}) > f(a_i)$. Choose $s = b_i$. Then

$$\left|\frac{g(t)-g(b_i)}{t-s}\right| \geq \frac{|g(t)-g(b_i)|}{2/3k} \geq \frac{1/m}{2/3k} = \frac{3k}{2m} > \frac{3mn}{2m} = \frac{3}{2}n > n.$$

Thus, $g \in U_n$ as claimed.

Claim III We will show each $\mathcal{U}_n$ is open. Let $f \in \mathcal{U}_n$. We will find a $\eta > 0$ s.t. $\|f-g\| < \eta \Rightarrow g \in \mathcal{U}_n$.

For each $t \in [0,1]$, $\exists s$ s.t. $\left|\frac{f(t)-f(s)}{t-s}\right| > n$.

We may regard s as a function of t and write $s(t)$.

$\exists\, \varepsilon(t)$ s.t. $\left|\frac{f(t)-f(s)}{t-s}\right| > n+\varepsilon$.

By continuity $\exists$ nbhd V_t of t s.t. $\forall t' \in V_t$ we have

$$\left|\frac{f(t')-f(s)}{t'-s}\right| > n+\varepsilon.$$

We can shrink V_t so that $s(t) \notin \overline{V}_t$. Do this for each $t \in [0,1]$. Then $\{V_t\}$ is an open covering of $[0,1]$. Let $\{V_{t_1}, U_{t_2}, \dots V_{t_k}\}$ be a subcover. Let...

$$\varepsilon = \min\{\varepsilon(t_i) : i=1\dots k\} > 0$$

$$\delta = \min\{dist(s(t_i), \overline{V}_{t_i}) : i=1,\dots k\} > 0$$

and $\eta = \frac{\varepsilon\delta}{2}$.

For any $t \in [0,1]$, $t \in V_{t_i}$ for some i. Let $s = s(t_i)$.

Then,

$$n+\varepsilon < \left|\frac{f(t)-f(s)}{t-s}\right| \leq \left|\frac{f(t)-g(t)}{t-s}\right| + \left|\frac{g(t)-g(s)}{t-s}\right| + \left|\frac{g(s)-f(s)}{t-s}\right|$$

We have $|f(t)-g(t)|$ and $|g(s)-f(s)|$ are $< \frac{\varepsilon\delta}{2}$,

and that $|t-s| \geq \delta$. Thus,

$$n+\varepsilon < \frac{\varepsilon}{2} + \left|\frac{g(t)-g(s)}{t-s}\right| + \frac{\varepsilon}{2}.$$

Thus,

$$\left|\frac{g(t)-g(s)}{t-s}\right| > n+\varepsilon-\varepsilon = n.$$

Hence, $g \in U_n$. $\blacksquare$