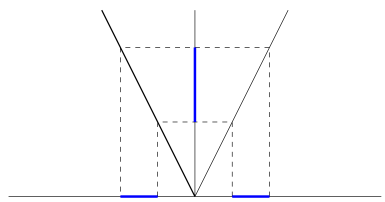
A Very Brief Intro to Ergodic Theory

Definition. Let X be a set, let $\mathcal{M}$ be a σ-algebra of subsets of X and let $m : \mathcal{M} \to [0, \infty]$ be a measure. If $m(X) = 1$ we call $(X, \mathcal{M}, m)$ a **probability space**.

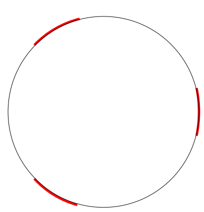
Definition. Let $T : X \to X$. If $m\left(T^{-1}(B)\right) = m(B)$ for all $B \in \mathcal{M}$, we say that T is **measure preserving**.

Example. Maps which rotate a circle are measure preserving. For $X = \mathbb{R}$ the function $T(x) = |2x|$ is measure preserving.



Definition. Let $(X, \mathcal{M}, m)$ be a probability space. A measure preserving function $T : X \to X$ is **ergodic** if the only sets $B \in \mathcal{M}$ with $T^{-1}(B) = B$ have $m(B) = 0$ or 1.

Example. Rotations of S^1 are ergodic iff they are irrational wrt 2π. The figure below, if rotated by $-2\pi/3$ would take the red set to itself, so a rotation by $2\pi/3$ is not ergodic. An irrational rotation of S^1, when iterated infinitely many times, can be shown to take any small open set and smear it all around S^1. Note: Often X is a compact topological group.



Theorem. The following are equivalent.

(a) T is ergodic.

(b) $\forall\ B \in \mathcal{M}$ with $m(B) > 0$ we have

$$m\left(\bigcup_{n=1}^{\infty} T^{-n}(B)\right) = 1.$$

(c) $\forall A, B \in \mathcal{M}$ with positive measures, $\exists\ n > 0$ such that

$$m\left(T^{-n}(A) \cap B\right) > 0.$$

Definition. Let $L^p(X, \mathcal{M}, m) =$ all measurable functions $f : X \to \mathbb{R}$ (or $\mathbb{C}$) such that

$$\int_X |f|^p < \infty.$$

Then $T : X \to X$ induces a map $U_T : L^p \to L^p$ via

$$U_T(f)(x) = f(T(x)).$$

Theorem. Let $f \in L^1$ and $f_n = \dfrac{1}{n}\displaystyle\sum_{i=0}^{n-1} f(T^i(x))$. Then

$$f_n \overset{\text{a.e.}}{\longrightarrow} f^* \in L^1, \quad f^* \circ T = f^*, \quad \& \quad \int_X f^* = \int_X f.$$

Furthermore, if T is ergodic, then f^* is a constant. Hence,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(T^i(x)) = \int_X f, \ \text{ for a.e. } x \in X.$$

Borel's Normal Number Theorem. For almost all numbers in [0,1) the frequency of 1's in the binary expansion is $\frac{1}{2}$.

Outline of proof. Let $T : [0,1) \to [0,1)$ be given by $T = 2x \bmod 1$. It is known that T is ergodic.

Let Y denote the set of points in $[0,1)$ that have a unique binary expansion. Since $[0,1) - Y$ is countable we have $m(Y) = 1$.

Let $x \in Y$ and write

$$x = \frac{a_1}{2} + \frac{a_2}{4} + \frac{a_3}{8} + \frac{a_4}{16} + \frac{a_5}{32} + \cdots$$

Then

$$T(x) = \frac{a_2}{2} + \frac{a_3}{4} + \frac{a_4}{8} + \frac{a_5}{16} + \frac{a_6}{32} + \cdots$$

Let $f(x) = \chi_{[\frac{1}{2},1)}(x)$. Then

$$f(T^i(x)) = f\left(\frac{a_{i+1}}{2} + \frac{a_{i+2}}{4} + \frac{a_{i+3}}{8} + \frac{a_{i+4}}{16} + \frac{a_{i+5}}{32} + \cdots\right) = a_{i+1}.$$

For $x \in Y$ the number of 1's in the first n digits is

$$\sum_{i=0}^{n-1} f\left(T^i(x)\right).$$

But,

$$\frac{1}{n}\sum_{i=0}^{n-1} f\left(T^i(x)\right) \xrightarrow{\text{a.e.}} \int_{[0,1)} \chi_{[\frac{1}{2},1)} = \frac{1}{2}.$$

$\square$

References or Further Reading

An Introduction to Ergodic Theory, by Peter Walters, Springer-Verlag, 1982.

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Normal Numbers are Normal, By Davar Khoshnevisan, *Clay Mathematics Institute Annual Report 2006*.

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