On The Second Proof of the Banach Contraction Principle

The textbook has a second proof of the Banach Contraction Principle, pg 241-242. I covered the first in class. The second proof looks easier, but that is because there is a gap in the proof. I'll go over this here:

Thm Let M be a complete metric sp. Let $f: M \to M$ be a be a contraction, i.e. $\exists k \in [0,1)$ s.t. $\forall x, y \in M$ $d(f(x), f(y)) \le k\, d(x,y)$. Then f has a unique fixed pt.

Pf 2 Uniqueness is proven the same way as proof 1.

Let $x_0 \in M$. Claim $\exists r_0 > 0$ s.t. $f\left(\overline{B(x_0, r_0)}\right) \subset B(x_0, r_0)$.
The book just assumes this, but it is not obvious.
To find r_0, let $y \in B(x_0, r_0)$. We need $d(x_0, f(y)) < r_0$.
We use the triangle inequality:

$$\begin{aligned} d(x_0, f(y)) &\le d(x_0, f(x_0)) + d(f(x_0), f(y)) \\ &\le d(x_0, f(x_0)) + k\, d(x_0, y) \\ &\le d(x_0, f(x_0)) + k\, r_0 \end{aligned}$$

Thus, we want to find $r_0 > 0$ s.t.

$$d(x_0, f(x_0)) + k r_0 < r_0 .$$

Thus, choose

$$r_0 > \frac{d(x_0, f(x_0))}{1-k},$$ which is

only zero if $f(x_0) = x_0$, and we are done, and is otherwise positive since $k < 1$.

From here you can follow the book's proof. W/ $B_0 = \overline{B(x_0, r_0)}$, $B_1 = f(\overline{B(x_0, r_0)})$, etc. you can check that $B_n \subset B_{n-1}$ and $\mathrm{diam}(B_n) \leq K^n \mathrm{diam}(B_0) \to 0$. Thus $\bigcap B_n$ = a single pt. $\{p\}$, $f(p) = p$ and if we let $x_n = f^{\circ n}(x_0)$, $x_n \to p$. ☒

Reference: I found the trick for getting r_0 from a ~~stundent'~~ student' paper: "The Banach Contraction Principle" by Alex Poniecki.

https://www.math.uchicago.edu/~may/VIGRE/VIGRE2009/REUP/Poniecki.pdf