The Riemann-Stieltjes Integral
(a brief introduction)

For more on this topic see, The Elements of Real Analysis, 2^{nd} ed, by Robert Bartle, pgs 212-259.

Def Let $f, g : [a,b] \to \mathbb{R}$ be bdd. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$, and let $T = \{t_1, t_2, \ldots, t_n\}$ be sample pts. The Riemann-Stieltjes sum of f wrt g is

$$S(P,T;f,g) = \sum_{k=1}^{n} f(t_k)\big(g(x_k) - g(x_{k-1})\big).$$

Def If $I \in \mathbb{R}$ is s.t. $\forall \varepsilon > 0$, $\exists P_\varepsilon$, a partition of $[a,b]$ with each $\Delta x < \varepsilon$, s.t. $\forall$ refinement P of P_ε, and any sample set T for P we have

$$|S(P,T;f,g) - I| < \epsilon,$$

then we say f is Riemann-Stieltjes integrable wrt g and write

$$I = \int_a^b f \, dg.$$

Ex Let $g(x) = \begin{cases} 0 & a \le x \le c \\ 1 & c < x \le b, \end{cases}$ and suppose f is continuous from the right at c. Then

$$\int_a^b f \, dg = f(c).$$

Pf Look at the partition members containing c.

Facts

1. $\displaystyle\int_a^b \alpha_1 f_1 + \alpha_2 f_2 \, dg = \alpha_1 \int_a^b f_1 \, dg + \alpha_2 \int_a^b f_2 \, dg.$

2. $\displaystyle\int_a^b f \, d(\alpha_1 g_1 + \alpha_2 g_2) = \alpha_1 \int_a^b f \, dg_1 + \alpha_2 \int_a^b f \, dg_2.$

3. If $g \in C'([a,b])$ and f is R-S int. wrt g, then
$$\int_a^b f \, dg = \int_a^b f g' \, dx.$$

4. $\displaystyle\int_a^b f \, dg + \int_a^b g \, df = f(x) g(x) \Big|_a^b.$

Def Regard $C^0([a,b])$ as a vector space and give it the sup. norm,

$$\|f\| = \sup_{x\in[a,b]} |f(x)|.$$

A *linear functional* is any linear function from $C^0([a,b])$ into $\mathbb{R}$.

A linear functional G is *positive* if $f(x) \geq 0 \Rightarrow G(f) \geq 0$.

A linear functional G is *bounded* if $\exists M \geq 0$ s.t.

$$|G(f)| \leq M\|f\|.$$

Fact If g is monotone increasing then $G(f) = \int_a^b f\,dg$ is a bdd. pos. lin. func.

Thm (*Riesz Repersentation Thm*) If $G: C^0([a,b]) \to \mathbb{R}$ is a bdd pos lin. func. then $\exists$ a monotone inc. func. $g: [a,b] \to \mathbb{R}$ s.t.

$$G(f) = \int_a^b f\,dg.$$