Proof that e is irrational

Reference: Principles of Mathematical Analysis by Rudin, pages 48-50.

Theorem: e is irrational.

Proof: Suppose e is rational and that $e = p/q$, where $p > 0$ and $q > 0$. In fact we can assume $q > 1$ since e is not an integer. (You should prove this!) We will derive a **contradiction**, namely that there is an integer between 0 and 1!

We know from Taylor's theorem that $e = \displaystyle\sum_{k=0}^{\infty} \frac{1}{k!}$. Let $e_n = \displaystyle\sum_{k=0}^{n} \frac{1}{k!}$.

Then $e - e_n = \dfrac{1}{(n+1)!} + \dfrac{1}{(n+2)!} + \dfrac{1}{(n+3)!} + \cdots$

$$< \frac{1}{(n+1)!}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots\right) =$$

$$\frac{1}{(n+1)!}\left(\frac{1}{1-\frac{1}{n+1}}\right) = \frac{1}{(n+1)!} \cdot \frac{n+1}{n+1-1} = \frac{1}{n!n}.$$

Thus, if we let $n = q$, we have $0 < e - e_q < \dfrac{1}{q!q}$.

Thus, $0 < q!(e - e_q) < \dfrac{1}{q} < 1$.

Now $q!e = \dfrac{q!p}{q} = (q-1)!p \in \mathbb{Z}$.

But also, $q!e_q = q!\left(1 + 1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \cdots + \dfrac{1}{q!}\right) \in \mathbb{Z}$.

Therefore, there exists an integer strictly between 0 and 1. Since this is absurd, we conclude that e cannot be expressed as a ratio of integers.