# Open Subsets of $\mathbb{R}$

**Definition.** $(-\infty, a)$, $(a, \infty)$, $(-\infty, \infty)$, $(a, b)$ are the *open intervals* of $\mathbb{R}$. (Note that these are the *connected* open subsets of $\mathbb{R}$.)

**Theorem.** Every open subset $U$ of $\mathbb{R}$ can be uniquely expressed as a countable union of disjoint open intervals. The end points of the intervals do not belong to $U$.

**Proof.** Let $U \subset \mathbb{R}$ be open. For each $x \in U$ we will find the "maximal" open interval $I_x$ s.t. $x \in I_x \subset U$. Here "maximal" means that for any open interval $J$ s.t $x \in J \subset U$, we have $J \subset I_x$.

Let $x \in U$. Define $I_x = (a_x, b_x)$, where $a_x = \inf \{a \in \mathbb{R} \,|\, (a, x) \subset U\}$, and $b_x = \sup \{b \in \mathbb{R} \,|\, (x, b) \subset U\}$. Either could be infinite. They are well defined since $\exists$ an open interval $I$ s.t $x \in I \subset U$, because $U$ is open.

Clearly, $x \in I_x \subset U$. Suppose $J = (p, q)$ is s.t. $x \in J \subset U$. Then $(p, x) \subset U$ so $p \geq a_x$. Likewise $q \leq b_x$. Thus, $J \subset I_x$ and so $I_x$ is indeed maximal.

Suppose $a_x \in U$. (In particular we are supposing $a_x$ is finite.) Then $\exists\ \epsilon > 0$ s.t. $(a_x - \epsilon, a_x + \epsilon) \subset U$. The $(a_x - \epsilon, b_x)$ is larger than $I_x$ contradicting maximality. Thus, $a_x \notin U$ and likewise $b_x \notin U$.

**Claim.** For $x, y \in U$, the intervals $I_x$ and $I_y$ are either disjoint or identical. **Proof.** If $I_x \cap I_y \neq \emptyset$, then $I_x \cup I_y$ is an open interval. Since $x \in I_x \cup I_y \subset U$, we have, by maximality, $I_x \cup I_y \subset I_x$. Likewise, $I_x \cup I_y \subset I_y$. Thus, by elementary set theory, $I_x = I_y$. □

We now have that $U$ is a disjoint union of maximal open intervals. Call this collection $\mathcal{I}$. How do we know there are at most countably many distinct members of $\mathcal{I}$? Remember that the rational numbers are a countable *dense* subset of $\mathbb{R}$.

For each distinct $I_x$ choose a rational point in $I_x$. Because these intervals are disjoint this determines a one-to-one map from $\mathcal{I}$ into $\mathbb{Q}$. Hence $\mathcal{I}$ is finite or countable. (See Chapter 1, Section 4 on cardinality.)

Now for uniqueness. Suppose $\mathcal{J}$ is a collection of disjoint open intervals whose union is $U$. Let $J = (a, b) \in \mathcal{J}$ and $x \in J$. We know there is an $I_x \in \mathcal{I}$. Clearly $I_x \cap J \neq \emptyset$. Since $I_x$ is maximal, $J \subset I_x$. We claim that $J = I_x$. Suppose not. Either $a_x < a$ or $b < b_x$. We assume the latter as both cases are similar. It follows that $b$ is finite. Now $b \notin J$, but $b \in I_x \subset U$. Thus, $\exists\ J' \in \mathcal{J}$ s.t. $b \in J'$. Now, $\exists \epsilon > 0$ s.t. $(b - \epsilon, b + \epsilon) \subset J'$. But then $\exists 0 < \delta < \epsilon$ s.t. $b - \delta$ is in $J$ and $J'$; hence they are not disjoint. Thus, $J = I_x$.

Now we have $\mathcal{J} \subset \mathcal{I}$. If $\mathcal{J} \neq \mathcal{I}$ then $\mathcal{J}$ cannot have union all of $U$. □

**Remark.** This result does not hold in all metric spaces. Indeed to make sense of it we would need a concept analogous to the open intervals. Even in $\mathbb{R}^2$ it is easy to draw open sets that are not the disjoint union of open balls. But, part of this result can be generalized to $\mathbb{R}^n$.

**Theorem.** Let $\mathcal{U}$ be a collection of disjoint open subsets of $\mathbb{R}^n$. Then $\mathcal{U}$ is at most countable.

**Outline of Proof.** Let $U \in \mathcal{U}$. Let $x \in U$. Let $B$ be an open ball s.t. $x \in B \subset U$. Show that $\exists\ y \in B$ with rational coordinates. This determines a one-to-one map from $\mathcal{U}$ into $\mathbb{Q}^n$, a countable set. $\square$

**Definition.** Let $M$ be a metric space. If $M$ contains a countable dense subset, we say $M$ is a *separable space* and the Theorem above holds. See pages 128-129 of the textbook. Later we will define what it means for a subset of a metric space to be *connected*. Then a further generalization is possible.