2.3 #8 Let γ be the line segment from 1 to i.

Evaluate $\int_\gamma \frac{1}{z}\,dz$.

Gamma is line segment from 1 to i.

<u>Method I</u> $\int_\gamma \frac{1}{z}\,dz = \log(i) - \log(1) = i\frac{\pi}{2}$, by Theorem 2.1.7.

<u>Method II</u> Let $\gamma(t) = (1-t) + it$, $0 \le t \le 1$.

$$\int_\gamma \frac{1}{z}\,dz = \int_0^1 \frac{-1+i}{1-t+it}\,dt = \int_0^1 \frac{-1+i}{1-t+it}\cdot\frac{1-t-it}{1-t-it}\,dt = (-1+i)\int_0^1 \frac{1-t-it}{(1-t)^2+t^2}\,dt$$

$$= (-1+i)\left[\int_0^1 \frac{1}{2t^2-2t+1}\,dt - \int_0^1 \frac{t}{2t^2-2t+1}\,dt - i\int_0^1 \frac{t}{2t^2-2t+1}\,dt\right]$$

$$= (-1+i)\underbrace{\int_0^1 \frac{1}{2t^2-2t+1}\,dt}_{A} + (-1+i)(-1-i)\underbrace{\int_0^1 \frac{t}{2t^2-2t+1}\,dt}_{B}$$

$$= (-1+i)A + 2B$$

Notice $2t^2-2t+1 = 2t^2-2t+\frac{1}{2}+\frac{1}{2} = \left(\sqrt{2}t - \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2} = \frac{1}{2}\left(2\left(\sqrt{2}t-\frac{1}{\sqrt{2}}\right)^2+1\right)$

$= \frac{1}{2}\left((2t-1)^2+1\right)$.

$$A = 2\int_0^1 \frac{1}{(2t-1)^2+1}\,dt = \int_{-1}^1 \frac{1}{u^2+1}\,du = \arctan(1) - \arctan(-1) = 2\arctan(1) = 2\cdot\frac{\pi}{4} = \frac{\pi}{2}.$$

Let $u = 2t-1$

Then $dt = \frac{1}{2}du$

$$B = 2\int_0^1 \frac{t}{(2t-1)^2+1}\,dt = \int_{-1}^1 \frac{\frac{u+1}{2}}{u^2+1}\,du = \frac{1}{2}\int_{-1}^1 \overbrace{\cancel{\frac{u}{u^2+1}}}^{0} + \frac{1}{u^2+1}\,du = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}.$$

Thus, $\int_\gamma \frac{1}{z}\,dz = (-1+i)\frac{\pi}{2} + 2\cdot\frac{\pi}{4} = -\frac{\pi}{2} + i\frac{\pi}{2} + \frac{\pi}{2} = i\frac{\pi}{2}$.