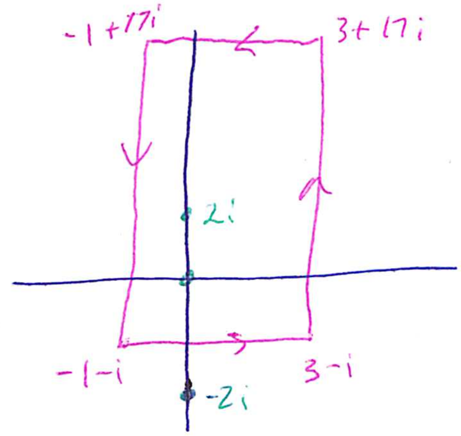
Example Compute $\displaystyle\int_\gamma \frac{(3+2i)z^2+12}{z^3+4z}\,dz$ where γ is the boundary of a rectangle with corners $(-1-i)$, $(3-i)$, $(3+17i)$, $(-1+17i)$, ccw



Solution

$$\frac{(3+2i)z^2+12}{z(z-2i)(z+2i)} = \frac{3}{z} + \frac{i}{z-2i} + \frac{i}{z+2i}$$

$$\int_\gamma \frac{3}{z}\,dz = 3(2\pi i) = 6\pi i$$

$$\int_\gamma \frac{i}{z-2i}\,dz = i(2\pi i) = -2\pi$$

$$\int_\gamma \frac{i}{z+2i}\,dz = i(0) = 0.$$

Thus, $\displaystyle\int_\gamma \frac{(3+2i)z^2+12}{z^3+4z}\,dz = \underline{-2\pi + 6\pi i}$

Example Compute $\int_\gamma \frac{z^2+e^z}{z(z-3)}\,dz$, when γ is (a) unit circle, center 0.
(b) circle radius 500, center 0.
(c) $\gamma(t)=10e^{it}-50i$, $0\le t\le 2\pi$.

Solution $\frac{1}{z(z-3)} = \frac{\frac{1}{3}}{z-3} - \frac{\frac{1}{3}}{z}$.

Let $f(z)=z^2+e^z$. $f(0)=1$, $f(3)=9+e^3$.

$$(\star) = \int_\gamma \frac{f(z)}{z(z-3)}\,dz = \frac{1}{3}\int_\gamma \frac{f(z)}{z-3}\,dz - \frac{1}{3}\int_\gamma \frac{f(z)}{z}\,dz$$

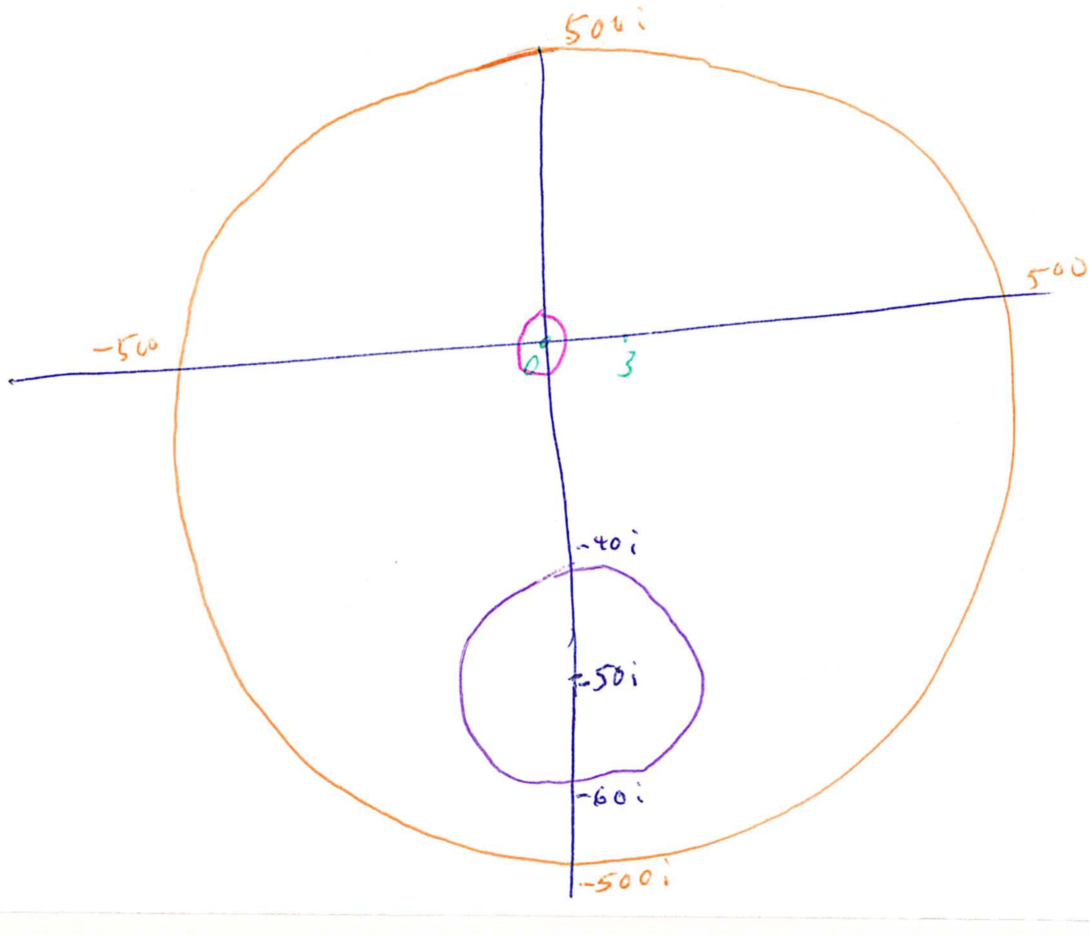
(a) $\int_\gamma \frac{f(z)}{z-3} = 0$ since 3 is outside γ. $\int_\gamma \frac{f(z)}{z}dz = 2\pi i\, f(0)\, I(\gamma,0) = 2\pi i$

Thus $(\star) = -\frac{1}{3}\cdot 2\pi i = \frac{-2\pi i}{3}$

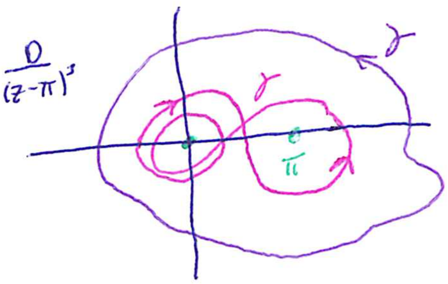
(b) $\int_\gamma f(z)\frac{dz}{z}$ is the same. $\int_\gamma \frac{f(z)}{z-3}\,dz = 2\pi i\, f(3)\cdot I(\gamma,3) = 2\pi i(9+e^3)$

Thus $(\star) = \frac{1}{3}(2\pi i)(9+e^3) - \frac{1}{3}(2\pi i) = \frac{2\pi(8+e^3)}{3}i$

(c) Both 0 and 3 are outside γ so, $(\star) = 0$.



Example Compute $\displaystyle\int_\gamma \frac{\cos^2 z}{z(z-\pi)^3}\,dz$ for each γ shown.



Solution Check that $\displaystyle\frac{1}{z(z-\pi)^3} = \frac{A}{z} + \frac{B}{z-\pi} + \frac{C}{(z-\pi)^2} + \frac{D}{(z-\pi)^3}$

where $A = \frac{-1}{\pi^3}$, $B = \frac{1}{\pi^3}$, $C = \frac{-1}{\pi^2}$, $D = \frac{1}{\pi}$.

$I(0,\gamma) = I(\pi,\gamma) = 1$. $I(0,\gamma) = -2$, $I(\pi,\gamma) = 1$.

Let $f(z) = \cos^2 z$. Then $f'(z) = 2\cos z \sin z = \sin 2z$ and $f''(z) = 2\cos 2z$.

Thus, $f(0) = 1$, $f(\pi) = 1$, $f'(0) = 0$, $f'(\pi) = 0$, $f''(0) = 2$, $f''(\pi) = 2$.

Now,
$$\int_\gamma \frac{\cos^2 z}{z}\,dz = 2\pi i\, f(0)\cdot I(0,\gamma) = \begin{cases} 2\pi i & \text{for } \gamma \\ -4\pi i & \text{for } \gamma \end{cases}$$

$$\int_\gamma \frac{\cos^2 z}{z-\pi}\,dz = 2\pi i\, f(\pi)\, I(\pi,\gamma) = 2\pi i \text{ for both curves.}$$

$$\int_\gamma \frac{\cos^2 z}{(z-\pi)^2}\,dz = 2\pi i\, f'(\pi)\, I(\pi,\gamma) = 0 \text{ for both curves.}$$

$$\int_\gamma \frac{\cos^2 z}{(z-\pi)^3}\,dz = \frac{2\pi i}{2!}\, f''(\pi)\, I(\pi,\gamma) = 2\pi i \text{ for both curves.}$$

Thus for γ we have
$$\int_\gamma \frac{\cos^2 z}{z(z-\pi)^3}\,dz = \frac{-2\pi i}{\pi^3} + \frac{2\pi i}{\pi^3} + 0 + \frac{2\pi i}{\pi} = \underline{2i}.$$

while for γ we get
$$\int_\gamma \frac{\cos^2 z}{z(z-\pi)^3}\,dz = \frac{+4\pi i}{\pi^3} + \frac{2\pi i}{\pi^3} + 0 + \frac{2\pi i}{\pi} = \frac{6i}{\pi^2} + 2i = \underline{\left(2 + \frac{6}{\pi^2}\right)i}$$

Note: I should have written $I(\gamma, x)$ instead of $I(x,\gamma)$.