1.1 **Complex Numbers**

Let $\mathbb{R}$ be the real numbers.
Let $\mathbb{R}^2 = \{(a,b) \mid a \in \mathbb{R}, b \in \mathbb{R}\}$.
Let $\mathbb{C} = \{a+bi \mid a \in \mathbb{R}, b \in \mathbb{R}, i \text{ is a symbol}\}$.

There is an obvious 1-to-1 correspondence between $\mathbb{R}^2$ and $\mathbb{C}$.

In $\mathbb{R}^2$ we have vector addition: $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$.

In $\mathbb{C}$ we have essentially the same addition operation:

$$a_1 + ib_1 + a_2 + ib_2 = a_1 + a_2 + i(b_1 + b_2).$$

But, we also have complex multiplication:

$$(a_1 + ib_1)(a_2 + ib_2) = a_1 a_2 - b_1 b_2 + i(a_1 b_2 + b_1 a_2).$$

In particular $i^2 = -1$.

Question: Can we divide complex numbers? Yes.

Suppose we are given $a+ib$, and we want to find $(x+iy)$ such that

$$(a+ib)(x+iy) = 1.$$

Assume $a+ib \neq 0$. Then we need to solve

$$ax - by = 1 \quad \text{and} \quad ay + bx = 0.$$

If $a=0$, use $x=0$, $y = \frac{1}{b}$. If $b=0$, use $x = \frac{1}{a}$, $y=0$.
Assume $a \neq 0, b \neq 0$.

$$\frac{a}{b}x - y = \frac{1}{b}, \quad y + \frac{b}{a}x = 0 \Rightarrow \frac{a}{b}x + \frac{b}{a}x = \frac{1}{b}$$

$$\Rightarrow \frac{a^2+b^2}{ab}x = \frac{1}{b} \Rightarrow x = \frac{a}{a^2+b^2}, \Rightarrow y = \frac{-b}{a}\left(\frac{a}{a^2+b^2}\right) = \frac{-b}{a^2+b^2}.$$

Thus, we define $\frac{1}{a+ib} = \frac{a}{a^2+b^2} + \frac{ib}{a^2+b^2}$. (This is consistent if $a=0$ or $b=0$.)

Thus, division is well defined. This make $\mathbb{C}$ into a field. See textbook page 6 for full definition.

Examples $(3+4i)(2-i) = 6+4+8i-3i = 10+5i$

$$\frac{1+i}{3-2i} = ? = \frac{1+i}{3-2i}\frac{(3+2i)}{(3+2i)} = \frac{3-2+3i+2i}{3^2+2^2} = \frac{1+5i}{13}$$

Trick!

Given them a few to do in class.

Other notation $\quad Re(a+ib) = a$
$\quad Im(a+ib) = b$

Let $z = a+ib$. Then $\overline{z} = a-ib$.
Also, $|z| = \sqrt{a^2+b^2}$. Notice: $|z|^2 = z\overline{z}$.

Square Roots We have $\sqrt{-1} = i$, since $i^2 = -1$ by definition. Notice $(-i)^2 = -1$ also. In general $\sqrt{-x} = i\sqrt{x}$, $x \in \mathbb{R}, >0$. But what about $\sqrt{a+ib}$? Proposition 1.1.3, pages 6-7, gives a for this:

$$\sqrt{a+ib} = \pm(\alpha + \mu\beta i) \quad \text{where}$$

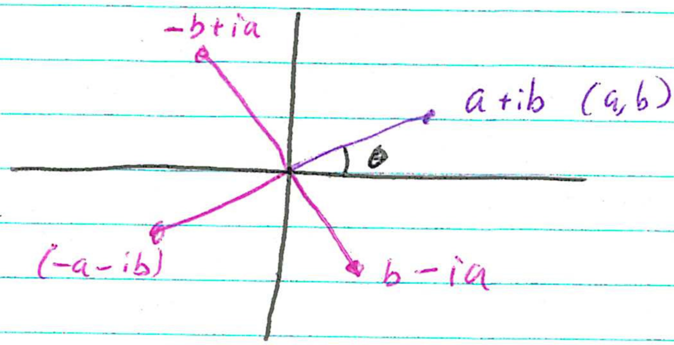
$$\alpha = \sqrt{\frac{a+\sqrt{a^2+b^2}}{2}}, \quad \beta = \sqrt{\frac{-a+\sqrt{a^2+b^2}}{2}} \quad \text{and}$$

$\mu = 1$ if $b \geq 0$ and $\mu = -1$ if $b < 0$. Read the proof on your ~~own~~ own. Do not memorize this formula, we will develop an easier method using trigonometry.

Fact If $a_n z^n + a_{n-1} z^{n-1} + \cdots a_1 z + a_0$ is an n degree poly. with a_i complex, then it has n roots, counting multiplicity in $\mathbb{C}$.

1.2 Properties of Complex Numbers. (Mostly Geometric)

Complex Plane



Notice, when you multiply $a+ib$ by -1, you rotate by $180°$ (π). When you multiple $a+ib$ by i you rotate by $90°$ ($\frac{\pi}{2}$), $i^2=-1$.

θ = argument of $a+ib$ = $\arg(a+ib)$.

If we want to restrict θ to be in $[0, 2\pi)$, then

$$\arg(a+ib) = \begin{cases} \arctan\left(\frac{b}{a}\right) & a>0 \quad b \geq 0 \\ \frac{\pi}{2} & a=0 \quad b>0 \\ \arctan\left(\frac{b}{a}\right)+\pi & a<0 \\ \frac{3\pi}{2} & a=0 \quad b<0 \\ \arctan\left(\frac{b}{a}\right)+2\pi & a>0 \quad b<0 \\ \text{undefined} & a=b=0. \end{cases}$$

Sometimes we think of $\arg(a+ib)$ as all $\{\theta + 2\pi n\}$, where θ is above. Some reference call the restricted version of arg as the principal value of arg and denote it by Arg. (Look up atan2.)

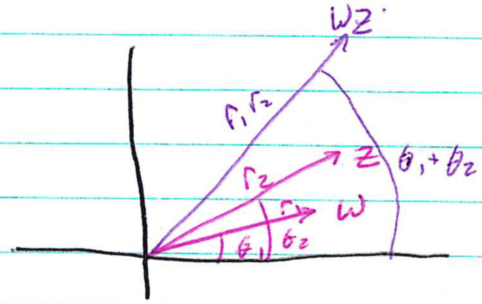
If $\theta = \arg a+ib$ we have $a = |a+ib| \cos\theta$,
$b = |a+ib| \sin\theta$.

Prop. 12.1 For complex numbers w, z we have

$$|wz| = |w||z| \quad \text{and} \quad \arg(wz) = \arg w + \arg z \pmod{2\pi}$$
(when defined)

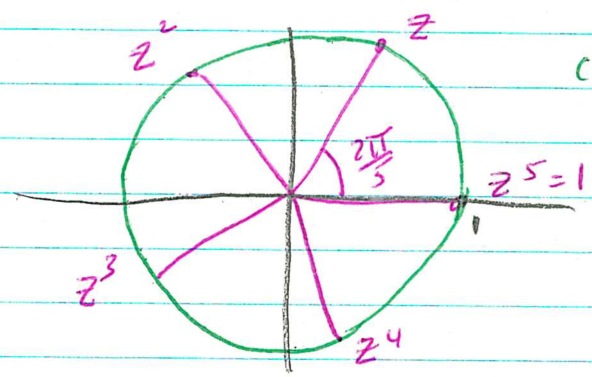
Pf Use trig. See textbook, pg 14.

This gives a nice way to see complex multiplication, much like we can see vector addition.



$w = r_1 \cos\theta_1 + i r_1 \sin\theta_1$
$z = r_2 \cos\theta_2 + i r_2 \sin\theta_2$
$wz = r_1 r_2 \cos(\theta_1+\theta_2) + i r_1 r_2 \sin(\theta_1+\theta_2)$.

Example Find a complex number $z \neq 1$ such that $z^5 = 1$.



circle of radius 1.

$\frac{2\pi}{5} = 72^\circ$

Let $z = \cos\theta + i\sin\theta$. Then $z^5 = \cos(5\theta) + i\sin(5\theta) = 1$. Thus, $5\theta = 2\pi n$. If $n = 0$, we get $z = \cos(0) + i\sin(0) = 1$. But if $n = 1$ we get

$$z = \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right).$$

In fact, there are five values of z with $z^5 = 1$.