1.3 Some Elementary Functions

I. Polynomials: see webpage (not in text)

II. Mobiüs Transformations. webpage (not in text)

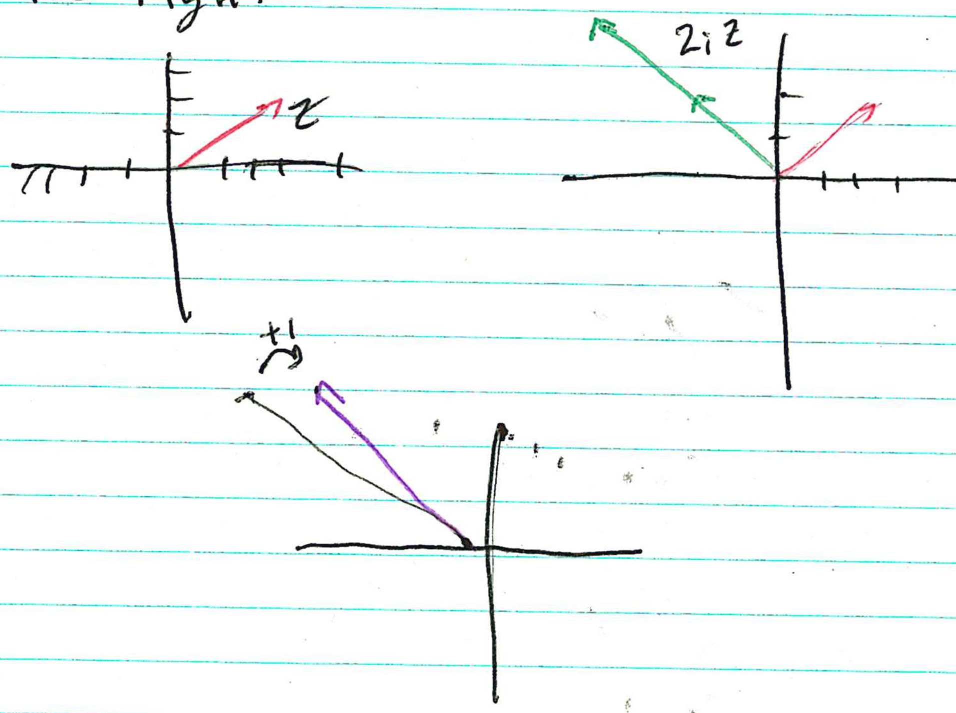
III exp, sin, cos, log

IV roots

0. How to visualized a $f: \mathbb{C} \to \mathbb{C}$? The graph would be 4-dimensional.

I. Poly. Example $f(z) = 2iz + 1$. We cannot graph it but we can understand it geometrically.

First rotate z (think of it as a vector) by 90° ccw. Then double its length, then shift one unit to the right.

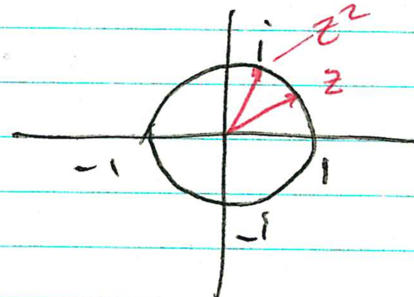


Example $f(z) = z^2$.

Feed in concentric circles with center $(0,0)$, $0+i0$.

If z is on unit circle, $|z|=1$, then so is z^2 since $|z^2| = |z|^2 = 1$. (Check this).

The argument, θ, with be doubled.



The image of the unit circle will be the unit circle, but the image will wrap around twice.

A circle of $R = \frac{1}{2}$ will be mapped to a circle of radius $\frac{1}{4}$, wrapped around twice.

A circle of $R = 2$, will be mapped to a circle of radius 4, wrapped around twice.

Etc.

Example z^3. Similar. ~~wrappes~~ wraps around 3 times.

On webpage I show images of several polynomials. Will discuss these in class.

II Möbius Transformations. $f(z) = \dfrac{az+b}{cz+d}$, $ad-bc \neq 0$

Think about $\frac{1}{z}$. Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Other examples on webpage.

On $\hat{\mathbb{C}}$ circles go to circles.

III The exponential function is very important.

Def For now we define $e^z = e^{x+iy} = e^x(\cos y + i \sin y)$.

It can also be defined using power series and the eq. above can be proven from this (see Ch. 3).

Notice: $e^{x+i0} = e^x(\cos(0) + i\sin(0)) = e^x$.

$$e^{i\pi} = e^0(\cos\pi + i\sin\pi) = -1.$$

$$e^{i\frac{\pi}{2}} = e^0\left(\cos\tfrac{\pi}{2} + i\sin\tfrac{\pi}{2}\right) = i$$

The graph of $e^{i\theta}$ in $\mathbb{C}$ over $\theta \in [0, 2\pi)$ is the unit circle, center 0.

Prop 1.3.2
(i) $e^{z+w} = e^z e^w$, $\forall z, w \in \mathbb{C}$.
(ii) $e^z \neq 0$ $\forall z \in \mathbb{C}$.
(iii) $x > 0 \Rightarrow e^x > 1$, $x < 0 \Rightarrow 0 < e^x < 1$ $(x \in \mathbb{R})$.
(iv) $|e^{x+iy}| = e^x$
(v) $e^{i\pi/2} = i$, $e^{i\pi} = -1$, $e^{i\frac{3\pi}{2}} = -i$, $e^{2\pi i} = 1$.
(vi) e^z is periodic $e^{z+2\pi n i} = e^z$. Period is $2\pi i$. $n \in \mathbb{Z}$.
(vii) $e^z = 1 \iff z = 2\pi n i$, $n \in \mathbb{Z}$

Some proofs.

(i) Use trig identities

(ii) For any $z \in \mathbb{C}$, $e^z \cdot e^{-z} = e^0 = 1$. Thus $e^z \neq 0$.

(iii) Can use Taylor series, $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$, for $x>0$ case. Since e^x is cont. and never zero, $e^x > 0$ $\forall x \in \mathbb{R}$. Since $e^0 = 1$ and e^x is increasing $(e^x)' = e^x > 0$, $e^x < 1$ for $x<0$.

(iv) $|e^{x+iy}| = |e^x(\cos y + i\sin y)| = e^x(\cos^2 y + \sin^2 y)^{1/2} = e^x$.

(v) you check.

(vi) ~~we did this~~ $e^{z+2\pi n i} = e^x(\cos(y+2\pi n) + i\sin(y+2\pi n))$
$= e^x(\cos y + i \sin y) = e^z$.

(vii) you check.

Fact $z = |z| e^{i\arg(z)}$, (often written as $z = re^{i\theta}$)

Just use definition.

Trig A little algebra show $\sin y = \dfrac{e^{iy} - e^{-iy}}{2i}$, $\cos y = \dfrac{e^{iy} + e^{-iy}}{2}$.

Compare to hyperbolic functions:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}.$$

These can be extended to complex inputs as well.

Prop 1.34 $\sin^2 z + \cos^2 z = 1$ $\forall z \in \mathbb{C}$.
$\sin(z+w) = \sin z \cos w + \cos z \sin w$, $\forall z, w \in \mathbb{C}$.
$\cos(z+w) = \cos z \cos w - \sin z \sin w$ $\forall z, w \in \mathbb{C}$.

Pf: see textbook.

Ex $\sin(i) = \dfrac{e^{ii} - e^{-ii}}{2i} = \dfrac{e^{-1} - e^{1}}{2i} = \dfrac{-i}{2}\left(\dfrac{1}{e} - e\right) = \dfrac{i}{2}\left(e - \dfrac{1}{e}\right)$

$\approx 1.1752\, i$

$\cos(i) = \dfrac{e^{ii} + e^{-ii}}{2} = \frac{1}{2}\left(e^{-1} + e\right) = \frac{1}{2}\left(e + \frac{1}{e}\right) \approx 1.54308$

Check that $\sin^2(i) + \cos^2(i) = 1$.

LOG

We want to define an inverse for $e^z = \exp(z) : \mathbb{C} \to \mathbb{C}$. Now e^z is not one-to-one, It has period $2\pi i$. We define a branch of the domain of e^z to be

$$A_{y_0} = \{x+iy \mid x \in \mathbb{R},\ y \in [y_0, y_0 + 2\pi)\}.$$

We will show e^z restricted to any A_{y_0} is one-to-one. We will also show the range is $\mathbb{C} - \{0\}$, so we will define $\text{Log} : \mathbb{C} - \{0\} \to A_{y_0}$.

Prop 1.3.5 ① e^z restricted to any A_{y_0} is one-to-one.
② The range of e^z on any A_{y_0} is $\mathbb{C} - \{0\}$.

Pf ① Suppose $e^{z_1} = e^{z_2}$. Then $e^{z_1 - z_1} = 1$.

By Prop. 1.3.3 (vii), $z_1 - z_2 = 2\pi n i$ for some n. But if both are in A_{y_0}, then $n = 0$ and $z_1 = z_2$.

② This is harder. Let $w \in \mathbb{C} - \{0\}$. We need to find $z \in A_{y_0}$ with $e^z = w$. Let $z = x+iy$, $w = u+iv$. Then, if $e^z = w$,

$$e^z = e^x e^{iy} = w, \implies |e^z| = |w| \text{ and } \arg e^z = \arg w.$$

$$\implies e^x = |w| \text{ and } e^{iy} = \frac{e^z}{e^x} = \frac{w}{|w|}.$$

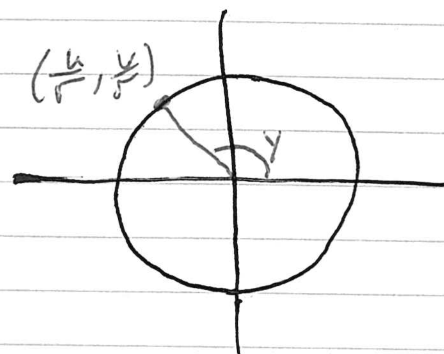
Thus, $x = \ln|w|$, where I'm using $\ln$ for the inverse of $e^x : \mathbb{R} \to \mathbb{R}^+$.

To solve $e^{iy} = \frac{w}{|w|}$ we review some trig.

We have $e^{iy} = \cos y + i\sin y = \frac{w}{|w|} = \frac{u}{\sqrt{u^2+v^2}} + \frac{iv}{\sqrt{u^2+v^2}}$.

So, $\cos y = \frac{u}{\sqrt{u^2+v^2}}$ and $\sin y = \frac{v}{\sqrt{u^2+v^2}}$.

Now $(u/\sqrt{\ }, v/\sqrt{\ })$ is a pt on the unit circle.



There is a unique value of y (up to 2π) that works.
Here is how to find $y \in [0, 2\pi)$.

Now $\tan y = \frac{v}{u}$ $(u \neq 0)$. $\tan^{-1}(\frac{v}{u}) \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

If $u = 0$, and $v > 0$, use $y = \frac{\pi}{2}$.
If $u = 0$ and $v < 0$, use $y = 3\pi/2$.

If $u > 0$, $v \geq 0$, then $y = \tan^{-1}(\frac{v}{u})$ will work.

If $u < 0$, then $y = \tan^{-1}(\frac{v}{u}) + \pi$ will work.

If $u > 0$, $v < 0$, then $y = \tan^{-1}(\frac{v}{u}) + 2\pi$ will work.

Once we have found a $y \in [0, 2\pi)$ we can add some $2\pi n$ to y to get $x + iy \in A_{y_0}$.

Notice: $y = \arg w$.

Def: $\log z = \ln|z| + i \arg z$, $\forall z \in \mathbb{C} - \{0\}$.

Examples

Compute $\log(i)$.

$$e^z = i$$
$$e^{x+iy} = i$$
$$e^x = 1 \Rightarrow x = 0$$
$$e^{iy} = i \Rightarrow y = \frac{\pi}{2} + 2\pi n$$

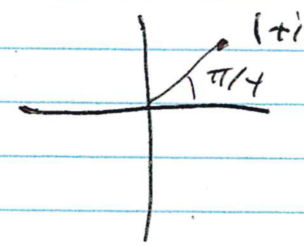
$$\log(i) = i\left(\frac{\pi}{2} + 2\pi n\right).$$

Compute $\log(-1)$.

$$e^z = -1$$
$$\Rightarrow e^{x+iy} = -1$$
$$e^x = |-1| = 1 \Rightarrow x = 0$$
$$e^{iy} = -1 \Rightarrow y = \pi + 2\pi n$$

$$\log(-1) = i(\pi + 2\pi n).$$

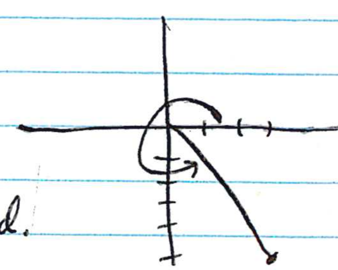
Compute $\log(1+i)$.



$|1+i| = \sqrt{2}$. $\quad \arg = \frac{\pi}{4}$.

$$\log(1+i) = \ln\sqrt{2} + i\left(\frac{\pi}{4} + 2\pi n\right)$$

Compute $\log(3-5i)$.

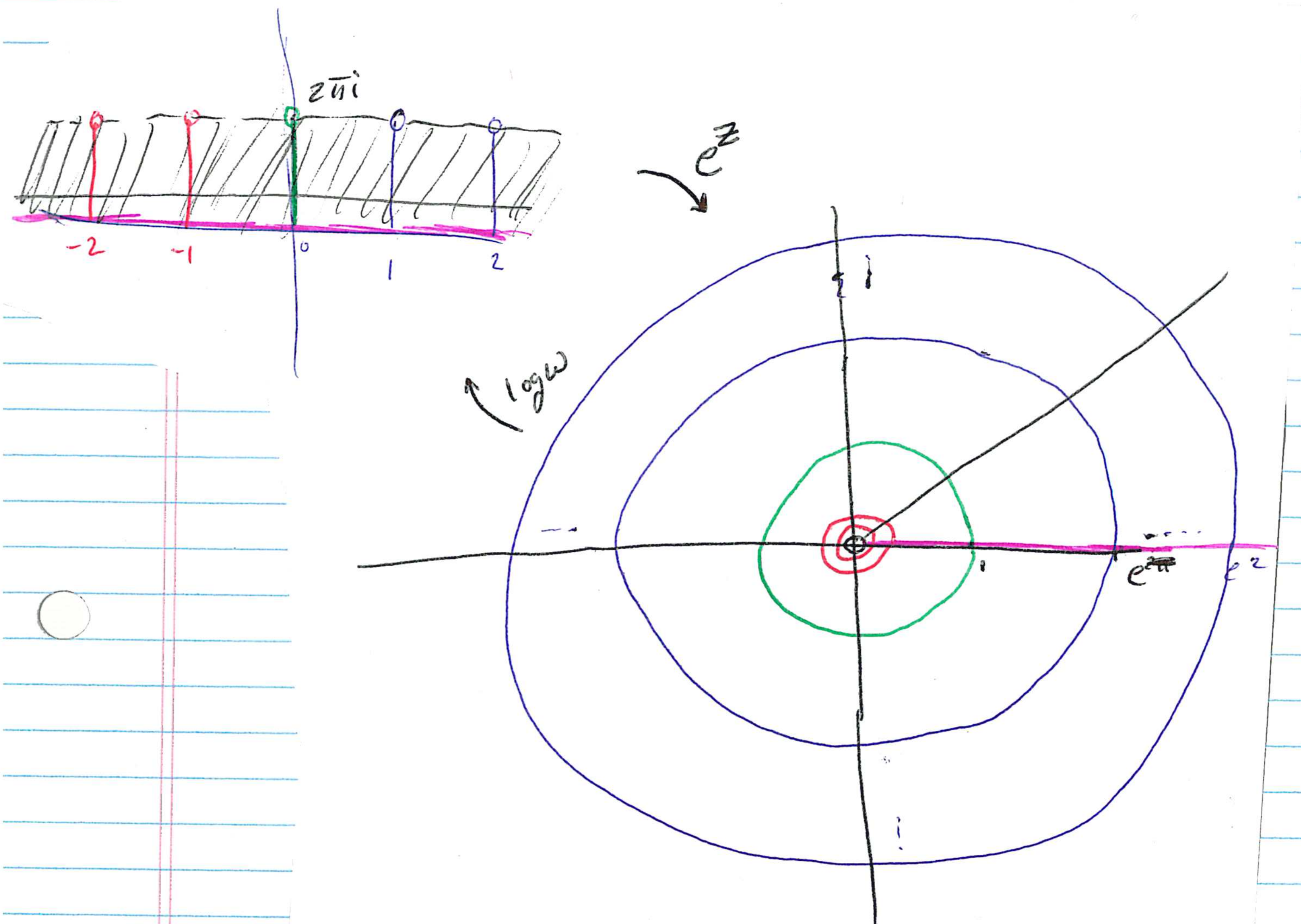


$$|3-5i| = \sqrt{3^2+5^2} = \sqrt{34}$$

Let $\theta = \tan^{-1}\left(-\frac{5}{3}\right) + 2\pi \approx 5.2528$ rad.

$$\log(3-5i) = \ln\sqrt{34} + i(\theta + 2\pi n)$$

Here is a graph of e^z on $\{x+yi \mid x \in \mathbb{R}, y \in [0, 2\pi)\}$.



Facts (Prop's 1.3.7, 1.3.8)

$$e^{\log z} = z \quad \text{and} \quad \log e^z = z \quad (\text{up to } 2\pi n i)$$

$$\log z_1 z_2 = \log z_1 + \log z_2 \quad (\text{up to } 2\pi n i) \quad z_1, z_2 \in \mathbb{C} - \{0\}.$$

Powers We have looked at integer powers of complex numbers. What if power is not an integer or is not real?

Def Let $a, b \in \mathbb{C}$, $a \neq 0$. Then we define

$$a^b = e^{b \log a}.$$

For b an integer we get a single value.
For b real and rational, say ~~s~~ $b = \frac{p}{q}$ is in reduced form we get q values.
Otherwise, we will have infinitely many values, one for each ~~branch~~ $n \in \mathbb{Z}$.

The textbook proves this in Prop. 1.3.10. I'll just do examples and you'll see how it works.

The first two examples use integer powers. You would not really do them this way!

Ex

$$(1+i)^2 = e^{2\log(1+i)} = e^{2(\ln\sqrt{2} + i\frac{\pi}{4} + 2\pi n i)}$$

$$= e^{\ln 2 + i\frac{\pi}{2} + 4\pi n i} = e^{\ln 2}\, e^{i\frac{\pi}{2}}\, \cancelto{1}{e^{4\pi n i}}$$

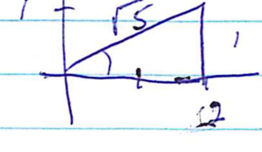
$$= 2i.$$

$$(1+i)^2 = 1^2 + 2\cdot 1\cdot i + i^2 = 1 + 2i - 1 = 2i.$$

Ex

$$(2+i)^{-1} = e^{-\log(2+i)} = e^{-(\ln\sqrt{5} + i\tan^{-1}(\frac{1}{2}) + 2\pi n i)}$$

$$= e^{-\ln\sqrt{5}}\, e^{-i\tan^{-1}(\frac{1}{2})} = \frac{1}{\sqrt{5}}\left(\cos(\tan^{-1}(\tfrac{1}{2})) - i\sin(\tan^{-1}(\tfrac{1}{2}))\right),$$



$$= \frac{1}{\sqrt{5}}\left(\frac{1}{\sqrt{5}} - \frac{2i}{\sqrt{5}}\right) = \frac{1}{5} - \frac{2}{5}i.$$

$$\frac{1}{2+i}\cdot\frac{2-i}{2-i} = \frac{2-i}{5}.$$

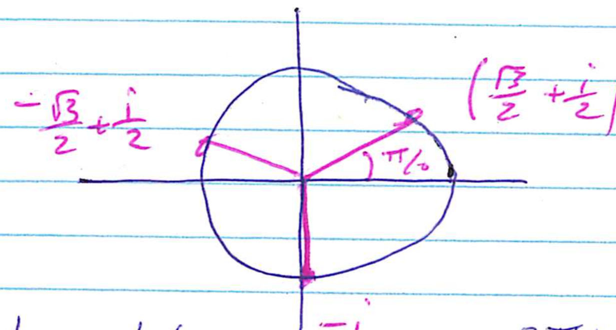
Ex $(i)^{\frac{1}{3}} = e^{\frac{1}{3}\log i} = e^{\frac{1}{3}\left(\frac{\pi}{2}i + 2\pi n i\right)} = e^{\frac{\pi}{6}i + \frac{2\pi n}{3}i}$

$n=0, \quad e^{\frac{\pi}{6}i} = \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{i}{2}$

$n=1, \quad e^{\frac{5\pi}{6}i} = \cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} + \frac{i}{2}$

$n=2, \quad e^{\frac{3\pi}{2}i} = \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) = 0 - i = -i.$

$n=3, \quad e^{\frac{15\pi}{6}i} = e^{\frac{\pi}{6}i}$ repeats. Only three roots.



The three cube roots of i.

Ex $(1)^{\frac{1}{5}} = e^{\frac{1}{5}\log 1} = e^{\frac{1}{5}(0 + 2\pi n i)} = e^{\frac{2\pi n i}{5}}$

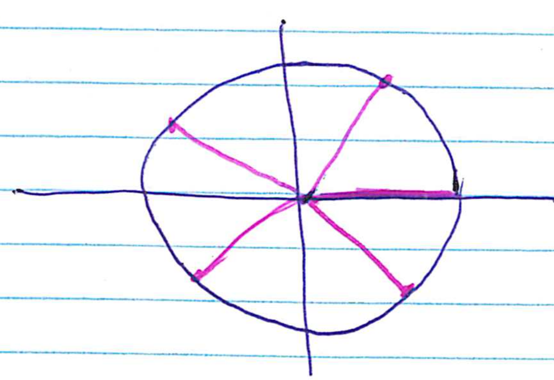
$n=0 \quad e^{0} = 1.$

$n=1 \quad e^{\frac{2\pi i}{5}} = \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right) \approx 0.31 + 0.95i$

$n=2 \quad e^{\frac{4\pi i}{5}} = \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right) \approx -0.81 + 0.59i$

$n=3 \quad e^{\frac{6\pi i}{5}} = \cos\left(\frac{6\pi}{5}\right) + i\sin\left(\frac{6\pi}{5}\right) \approx -0.81 - 0.59i$

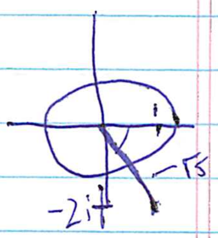
$n=4 \quad e^{\frac{8\pi i}{5}} = \cos\left(\frac{8\pi}{5}\right) + i\sin\left(\frac{8\pi}{5}\right) \approx 0.31 - 0.95i$



The five fifth roots of unity - one.

Ex Find $(1-2i)^{\frac{3}{4}}$. We will find the four fourth roots of $(1-2i)$ and then cube them.

$$(1-2i)^{\frac{1}{4}} = e^{\frac{1}{4}\log(1-2i)} = e^{\frac{1}{4}(\ln\sqrt{5} + i\tan^{-1}(-2) + 2\pi n i)}$$



$$= 5^{\frac{1}{8}} e^{-\frac{i}{4}\tan^{-1}(2) + \frac{\pi n}{2} i}$$

$\theta = \tan^{-1}\left(\frac{-2}{1}\right)$

$n=0$: $5^{\frac{1}{8}} e^{-\frac{i\tan(2)}{4}} = 5^{\frac{1}{8}}\left(\cos\left(\frac{\tan^{-1}(2)}{4}\right) - i\sin\left(\frac{\tan^{-1}(2)}{4}\right)\right)$

$\approx 1.1763 - 0.3342i$

$n=1$: $5^{\frac{1}{8}} e^{i\left(-\frac{\tan^{-1}(2)}{4} + \frac{\pi}{2}\right)} = 5^{\frac{1}{8}}\left[\cos\left(\frac{\pi}{2} - \frac{\tan^{-1}2}{4}\right) + i\sin\left(\frac{\pi}{2} - \frac{\tan^{-1}(2)}{4}\right)\right]$

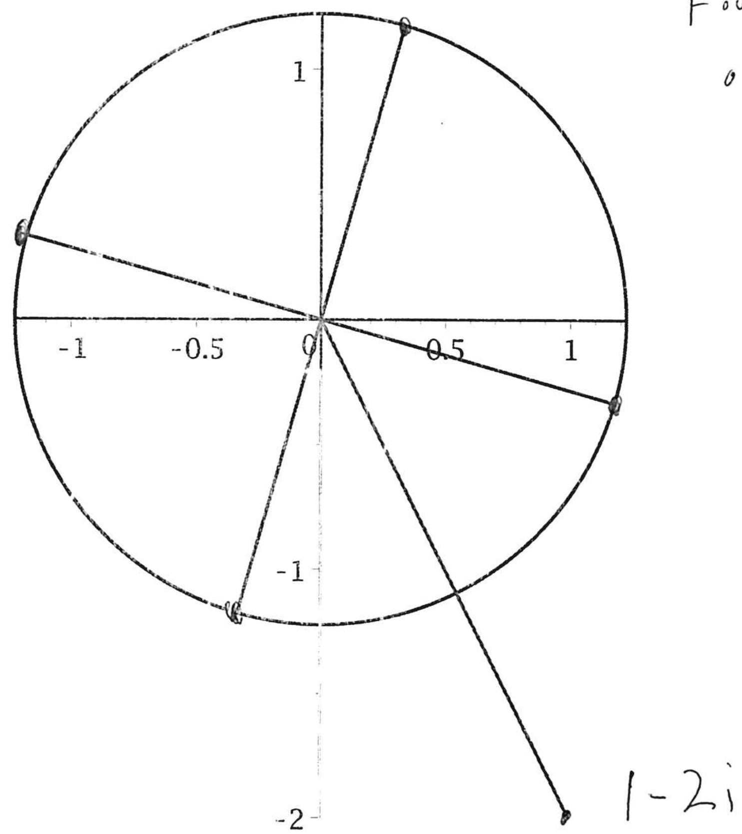
$\approx 0.3342 + 1.1763i$

$n=2$: $5^{\frac{1}{8}} e^{i\left(\frac{-\tan^{-1}(2)}{4} + \pi\right)} \approx -1.1763 + 0.3342i$

$n=3$: $5^{\frac{1}{8}} e^{i\left(\frac{-\tan^{-1}(2)}{4} + \frac{3\pi}{2}\right)} \approx -0.3342 - 1.1763i$

See plot in next page. See website for this plot and the cubes of these four points.

Four Fourth Roots
of 1-2i.



Ex $1^i = e^{i \log(1)} = e^{i(\ln 1 + 2\pi n i)} = e^{-(2\pi n)}$

$n=0$, $e^0 = 1$. $n=1$, $e^{-2\pi} \approx 0.001867$

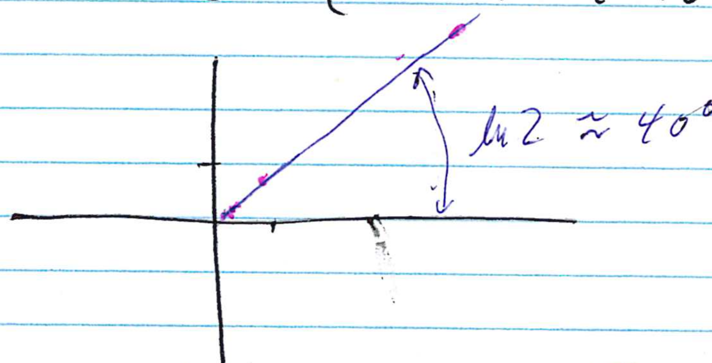
$n=2$, $e^{-4\pi} \approx 0.000003487$ etc.

$n=-1$, $e^{2\pi} \approx 535.5$, $n=-2$, $e^{4\pi} \approx 286{,}751.3$ etc.

Ex $2^i = e^{i \log(2)} = e^{i(\ln 2 + 2\pi n i)} = e^{-2\pi n} e^{i \ln 2}$

$= e^{-2\pi n}(\cos \ln 2 + i \sin \ln 2)$

$\approx e^{-2\pi n}(0.7692 + 0.6390 i)$



Ex $(i)^i = e^{i \log i} = e^{i(\frac{\pi}{2} i + 2\pi n i)} = e^{-\frac{\pi}{2} - 2\pi n} \in \mathbb{R}$.

Ex $(1)^e = e^{e \log(1)} = e^{e(0 + 2\pi n i)} = e^{2\pi e n i}$, $n \in \mathbb{Z}$.

Compute a few values if you like. They form a dense subset of the unit circle.