Section 1.4
Continuity

Open Sets. Let $D(z,r) = \{w \in \mathbb{C} \,|\, |z-w| < r\}$. It is called the **open disk** with center z and radius r. A subset $U \subset \mathbb{C}$ is open if $\forall\ z \in U$, $\exists\ r > 0$ s.t. $D(z,r) \subset U$. It can be shown that an open disk is open!

Informally we might say U does not contain any points in its boundary. An equivalent definition is that U is open if it is a union of open disks.

In $\mathbb{R}$ the set $(0,1)$ is open. The set $\{x+iy \,|\, 0 < x < 1 \,\&\, y = 0\}$ is not open in $\mathbb{C}$. Th set $\{x+iy \,|\, 0 < x < 1 \,\&\, 0 < y < 1\}$ is open in $\mathbb{C}$.

The textbook calls an open disk $D(z,r)$ a **neighborhood** of z. Other books call any open set containing z a neighborhood of z.

A **deleted neighborhood** of z is a set of the form

$$\{w \in \mathbb{C} \,|\, 0 < |z-w| < r\} = D(z,r) - \{z\}.$$

Closed Sets. A subset $C \subset \mathbb{C}$ is **closed** if $\mathbb{C} - C$ is open.
Example. $\mathbb{R}$ is closed in $\mathbb{C}$.

Facts.

(a) $\mathbb{C}$ and $\emptyset$ are open and closed! Easy to check.
(b) The union of any collection of open sets is open.
(c) The intersection of any finite collection of open sets is open.
(d) The intersection of any collection of closed sets is closed.
(e) The union of any finite collection of closed sets is closed.

Proof of (b). Let J be any set. (It is called an index set, it could be finite or infinite.) Let $\{U_\alpha \,|\, \alpha \in J\}$ be a collection of open subsets of $\mathbb{C}$. Let

$$U = \bigcup_{\alpha \in J} U_\alpha.$$

Let $z \in U$. For at least one $\alpha' \in J$, we will have $z \in U_{\alpha'}$. Thus, $\exists\ r > 0$ s.t. $D(z,r) \subset U_{\alpha'}$. But, $U_{\alpha'} \subset U$. Thus, $D(z,r) \subset U$. We conclude that U is open.

Proof of (c). Let A and B be open. If $A \cap B = \emptyset$, we are done by (a). Suppose $z \in A \cap B$.
$\exists\ r_1 > 0$ s.t. $D(z,r_1) \subset A$.
$\exists\ r_2 > 0$ s.t. $D(z,r_2) \subset B$.
Let r be the smaller of r_1 and r_2.
Then $D(z,r)$ is in both $D(z,r_1)$ and $D(z,r_2)$.
Hence $D(z,r)$ is in both A and B, which is to say $D(z,r) \subset A \cap B$.
Thus, $A \cap B$ is open by definition.

Suppose $\{U_i \,|\, i = 1, \ldots, n\}$ are open. We know $U_1 \cap U_2$ is open. If $U_1 \cap \cdots \cap U_k$ is open, then so is

$$(U_1 \cap \cdots \cap U_k) \cap U_{k+1}.$$

Thus, by the Principle of Mathematical Induction, $U_1 \cap \cdots \cap U_n$ is open for any finite natural number n.

Note. The intersection of an infinite collection of open sets need not be open. For example, let $U_n = D(0, \frac{1}{n})$ for $n = 1, 2, 3, \ldots$. Then

$$\bigcap_{n=1}^{\infty} = \{0\},$$

which is not open.

Proof of (d). Let $\{C_\alpha \,|\, \alpha \in J\}$ be closed subsets of $\mathbb{C}$.
Let $U_\alpha = \mathbb{C} - C_\alpha$ for each α in J. These are open.
Let $U = \bigcup\limits_{\alpha \in J} U_\alpha$. Then U is open by (b).
Let $C = \mathbb{C} - U$. Then C is closed by definition.
But, $C = \bigcap\limits_{\alpha \in J} C_\alpha$. Think about that!

Proof of (e). Let A and B be closed. Let $U = \mathbb{C} - A$ and $V = \mathbb{C} - B$. They are open. Thus, $W = U \cap V$ is open. Let $C = \mathbb{C} - W$. Then C is closed. But, $C = A \cup B$. Continue by induction.

Note. Infinite unions of closed sets need not be closed. For example, let $C_n = \{z \in \mathbb{C} \,|\, |z| \leq \frac{1}{n}\}$, for $n = 2, 3, \ldots$. Then

$$\bigcup_{n=2}^{\infty} C_n = \{z \in \mathbb{C} \,|\, |z| < 1\} = D(0, 1) \subset \mathbb{C}.$$

Limits of Sequences.

A **sequence** in $\mathbb{C}$ is an infinite list of elements of $\mathbb{C}$.

$$(z_1, z_2, z_3, \ldots) = (z_n)_{n=1}^{\infty}$$

We say $\lim\limits_{n\to\infty} z_n = z$, or $z_n \to z$, if $\forall\ \epsilon > 0$, $\exists\ N$ s.t. $n \geq N \implies |z - z_n| < \epsilon$ or equivalently $z_n \in D(z, \epsilon)$.

These properties hold: If $a \in \mathbb{C}$, $z_n \to z$ and $w_n \to w$ then

- $az_n \to az$.
- $z_n + w_n \to z + w$.
- $z_n w_n \to zw$.
- $1/z_n \to 1/z$ as long as $z \neq 0$ and no $z_n = 0$.

Proofs are the same as in Calculus. They are covered in detail in MATH 352.

Fact. Limits are unique. If $z_n \to z$ and $z_n \to w$, then $z = w$.
Proof: Let $\epsilon > 0$. Let N be s.t. $n \geq N \implies |z_n - z| < \epsilon$ and $|w_n - z| < \epsilon$. Then for $n \geq N$

$$|z - w| = |z - z_n + z_n - w| \leq |z - z_n| + |z_n - w| < 2\epsilon.$$

This means the distance between z and w is smaller than every positive number. So, it must be zero. Hence, $z = w$.

Fact. If $(z_n) \subset C$, a closed subset of $\mathbb{C}$, and $z_n \to z$, then $z \in C$.
Proof: Suppose this is false. Then $z \in \mathbb{C} - C$, an open set. Choose $\epsilon > 0$ s.t. $D(z, \epsilon) \subset \mathbb{C} - C$. But then $|z_n - z| \geq \epsilon$ for all n.

A sequence (z_n) is **Cauchy** if $\forall\ \epsilon > 0$, $\exists\ N$ s.t. $m, n \geq N \implies |z_m - z_n| < \epsilon$.

Fact. Cauchy sequences converge.
Proof: Done in Math 352 for real sequences. The proof is the same.

This fails is some spaces. A sequence of rational numbers can be Cauchy, but the limit might not be rational.

Limits of Functions.
Let $f : D \to \mathbb{C}$ where $D \subset \mathbb{C}$. Let $z_o \in \mathbb{C}$ and assume $\exists\ r > 0$ s.t. $D(z_o, r) - \{z_o\} \subset D$. (The point z_o may or may not be in the domain D of the function f.) Then

$$\lim_{z \to z_o} f(z) = L$$

means $\forall\ \epsilon > 0$, $\exists\ \delta > 0$ s.t. $0 < |z - z_o| < \delta \implies |f(z) - L| < \epsilon$.

Facts. Limits are unique. If $\lim_{z\to z_o} f(z) = L$ and $\lim_{z\to z_o} f(z) = K$, the $L = K$.
Proof: See textbook.

Suppose, $a \in \mathbb{C}$, $\lim_{z\to z_o} f(z) = L$ and $\lim_{z\to z_o} g(z) = K$. Then

- $\lim_{z\to z_o} af(z) = aL$
- $\lim_{z\to z_o} f(z) + g(z) = L + K$
- $\lim_{z\to z_o} f(z)g(z) = LK$
- $\lim_{z\to z_o} 1/g(z) = 1/K$, provide $K \neq 0$ and $g(z) \neq 0$ on a deleted nbhd of z_o.

Proof: See textbook. Covered in MATH 352 for real cases. Proofs are the same.

Continuity!

Let $A \subset \mathbb{C}$ and $f : A \to \mathbb{C}$. There are two equivalent definition of "f is continuous on A".

1. Let $z_o \in A$. If $\displaystyle\lim_{z\to z_o} f(z) = f(z_o)$, then f is continuous at z_o. If this holds for all $z_o \in A$, then f is continuous on A.

2. Let $z_o \in A$. If $\forall\, \epsilon > 0$, $\exists\, \delta > 0$ s.t. $|z - z_o| < \delta$ and $z \in A \implies |f(z) - f(z_o)| < \epsilon$.

The proof that these are equivalent just involves unpacking the definition a limit.

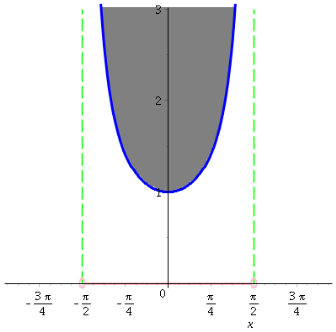
Facts. Sums and products of continuous functions are continuous. The reciprocal of a continuous function is continuous except at points where the function was zero. The composition of continuous functions are continuous when defined. Proofs can be done using the facts about limits.

Theorem. Let $A \subset \mathbb{C}$ and $f : A \to \mathbb{C}$. (a) The function f is continuous on A if and only if the inverse image of open sets are open relative to A. (b) The function f is continuous on A if and only if the inverse image of closed sets are closed relative to A.

A set $B \subset A$ is open/closed relative to A if there is an open/closed set D s.t. $B = A \cap D$.

The continuous image of an open set need not be open. For example, let $f(z) = 5 + 2i$. Then the image of any open set is $\{5 + 2i\}$, which is not open.

The continuous image of an close set need not be closed. For example, let $f(x+iy) = x$. Let $D = \big\{x + iy \,|\, -\frac{\pi}{2} < x < \frac{\pi}{2} \,\&\, y \geq \sec x\big\}$. Then $f(D) = \big\{x + iy \,|\, -\frac{\pi}{2} < x < \frac{\pi}{2} \,\&\, y = 0\big\}$. You can check that D is closed but $f(D)$ is not. See figure below.



The proof of this theorem is optional reading. It is covered in MATH 530.

Proof. I'll insert the proof here later for those interested. ☐

Compact Sets. There are three standard equivalent definitions for a set to be compact in $\mathbb{C}$.

(1) $K \subset \mathbb{C}$ is compact if it is closed and bounded. (Bounded means $\exists\, R > 0$ such that $K \subset D(0, R)$.)
(2) $K \subset \mathbb{C}$ is compact if every sequence of points in K has a subsequence that converges (the limit has to be in K).
(3) $K \subset \mathbb{C}$ is compact if every covering of K by open sets has a finite subcollection that covers K. This means $\forall$ collection of open sets whose union contains K, $\exists$ a finite subset of that collection whose union still contains K.

The proofs that these are indeed equivalent are covered in MATH 530. The same definitions are used to define compact subsets of $\mathbb{R}$.

Fact (Proposition 1.4.19). Let $K \subset \mathbb{C}$ be compact. Let $f : K \to \mathbb{C}$ be continuous. Then $f(K)$ is compact in $\mathbb{C}$.

Proof: Let (z_n) be a sequence in $f(K)$, For each n let $w_n \in f^{-1}(z_n)$. Since K is compact $\exists$ a subsequence, (w_{n_i}) that converges as $i \to \infty$ to some $w \in K$. But then (z_{n_i}) converges to $f(w) \in f(K)$ since f is continuous. Hence $f(K)$ is compact.

Note. Let $A \subset \mathbb{C}$ and let $f : A \to \mathbb{R}$. Limits and continuity are defined just the same.

Extreme Value Theorem (1.4.20). Let $K \subset \mathbb{C}$ be compact. Let $f : K \to \mathbb{R}$ be continuous. Then $\exists$ points a and b in K s.t. $f(a) \leq f(z) \leq f(b)$ for all $z \in K$. In words, $f(z)$ has minimum and maximum values. Note: a and b need not be unique or distinct.

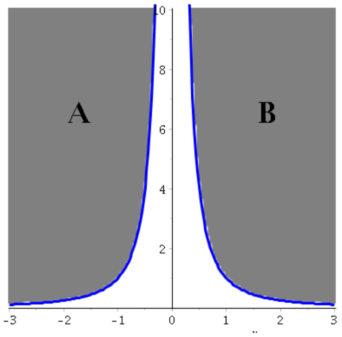
Proof: $f(K)$ is closed and bounded in $\mathbb{R}$.

The **distance** between to sets A and B in $\mathbb{C}$ is defined to be

$$\mathrm{dist}(A, B) = \inf\{|z - w| \mid z \in A \,\&\, w \in B\}.$$

If $A \cap B \neq \emptyset$, then $\mathrm{dist}(A, B) = 0$. If A and B are disjoint it is possible for the distance between them to be zero. For example if A is $\{x + iy \mid y > 0\}$ and B is $\{x + iy \mid y < 0\}$ then $A \cap B = \emptyset$ and $\mathrm{dist}(A, B) = 0\}$. The figure below shows two closed disjoint sets with distance zero. We let

$$A = \{x + iy \mid x < 0, y > 1/x^2\} \;\;\&\;\; B = \{x + iy \mid x > 0, y > 1/x^2\}.$$



Lemma 1.4.21. If A is compact and B is closed and they are disjoint, then $\mathrm{dist}(A, B) > 0$.

Proof: See textbook.

Uniform Continuity. Let $A \subset \mathbb{C}$ and let $f : A \to \mathbb{C}$. Then f is **uniformly continuous** on A if $\forall\, \epsilon > 0$, $\exists\, \delta > 0$ s. t. $|z - w| < \delta \implies |f(z) - f(w)| < \epsilon$, $\forall\, z$ and w in A.

The point is given an $\epsilon > 0$ one choice of $\delta > 0$ work throughout the set A.

Quick Examples of $\mathbb{R} \to \mathbb{R}$ functions.

- x^2 on $\mathbb{R}$ is continuous but not uniformly continuous.
- $1/x$ on $(0, \infty)$ is continuous but not uniformly continuous.
- $\arctan x$ is uniformly continuous on $\mathbb{R}$.

Theorem. If $f : A \to \mathbb{C}$ is continuous on A and A is compact, then f is uniformly continuous on A.

Proof: See textbook.

Connected Sets. Let $A \subset \mathbb{C}$. Suppose we can find two open sets, U and V, s. t.

$$U \cap V = \emptyset \quad A \subset U \cup V \quad A \cap U \neq \emptyset \quad A \cap V \neq \emptyset.$$

Then we say U and V form a **separation** of A.

If there is no separation of A we say that A is a **connected** set. If there is a separation of A we say that A is a **disconnected** set.

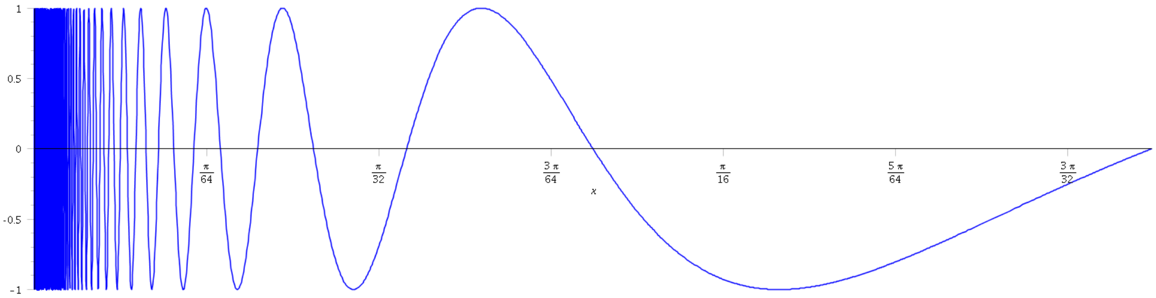
Essentially the same definitions are used for subsets of $\mathbb{R}$.

Path Connected Sets. Let $A \subset \mathbb{C}$. A **path** in A is a continuous function $\gamma : [0,1] \to A$. The point $\gamma(0)$ is the starting point of the path and $\gamma(1)$ is the ending point. We say γ is a path from $\gamma(0)$ to $\gamma(1)$ lying in A. If for any two points z and w is A, $\exists$ a path from z to w lying in A, we say that A is path connected.

Facts. A path connected set is connected. But a connected set need not be not be path connected. However, an open connected set is path connected.

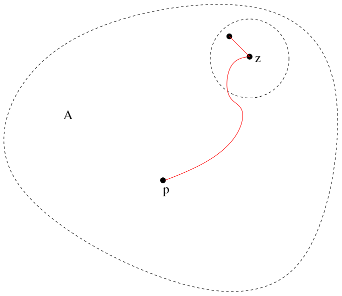
Proof that path connect sets are connected. We shall assume $[0,1]$ is connected. This fact to proven in courses of real analysis and topology (MATH 452 and MATH 530). The proof is by contradiction. Suppose A is a set which is path connected but not connected. Let U, V be a separation of A. Let $p \in A \cap U$ and $q \in A \cap V$. Let $\gamma(t)$ be a path from p to q is A. Then the sets $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$ form a separation of $[0,1]$.

Example showing a connected set need not be path connected. Let $S = \{x+iy \,|\, 0 < x \leq 1/\pi,\ y = \sin(\frac{1}{x}\}$ and $Y = \{iy \,|\, -1 \leq y \leq 1\}$. Then S and Y are path connected and connected, while the set $S \cup Y$ is connected but not path connected. This is proven in topology textbooks. See figure below.



Proof that an open connected set is path connected. Let A be an open set which is not path connected. We will show it cannot be connected. Since A is not path connected there exists a pair of points, p and q, such that there is no path from p to q. Let U be all the points in A for which there is a path from p and let $V = A - U$. We claim U, V forms a separation of A. It is clear that $A = U \cup V$, $\emptyset = U \cap V$, $p \in A \cap U \implies A \cap U \neq \emptyset$, and $q \in A \cap V \implies A \cap V \neq \emptyset$. It only remains to show that U and V are open. Let $z \in U$. Then $z \in A$. Since A is open $\exists\ r > 0$ s.t. $D(z,r) \subset A$. There is a path in $D(z,r)$ from z to any other point in $D(z,r)$. Thus, we can form a path from p to any point in $D(z,r)$. (See figure, this is proven formally in topology courses using something called the Pasting Lemma.) Thus, $D(z,r) \subset U$ and hence U is open. Now let $z \in V$. Since A is open $\exists\ r > 0$ s.t. $D(z,r) \subset A$. There is a path in $D(z,r)$ from z to any other point in $D(z,r)$. Let $w \in D(z,r) - \{z\}$ Therefore, if there was a path from p to w lying in A, there would be a path from p to z lying in A. But, this is not possible, therefore there is no path from p to w in A. Hence, $D(z,r) \subset V$ and so V is open. Thus, A is disconnected. It follows that if A was connected it would have to be path connected.

Note: This can be done in such a way that the path is is differentiable. See Proposition 1.4.15, page 49.



Facts. Let $f : A \to \mathbb{C}$ be continuous. If A is connected, so is $f(A)$. If A is path connected, so is $f(A)$.

Proof. Suppose, $f : A \to \mathbb{C}$ is continuous, A is connected, but $f(A)$ is disconnected. Let U, V be a separation of $f(A)$. Then you can show $f^{-1}(U)$, $f^{-1}(V)$ gives a separation of A. Since this is impossible, it must have been that $f(A)$ was connected.

Suppose, $f : A \to \mathbb{C}$ is continuous and A is path connected. Let p and q be points in $f(A)$. Let $a \in f^{-1}(p)$ and $b \in f^{-1}(q)$. Then there is a path $\gamma : I \to A$ from a to b in A. Then $f \circ \gamma$ is a path in $f(A)$ from p to q. Thus, $f(A)$ is path connected.

Path Covering Lemma.

In Chapter 2 we will be studying integration along paths in $\mathbb{C}$. The Path Covering Lemma is a technical result that is used proving properties of such integrals. Let $\gamma : [a, b] \to G \subset \mathbb{C}$, where G is open and γ is continuous. It will be helpful to find a finite collection, D_1, D_2, $\ldots$, D_n of open disks in G with centers on the path such each contains the center points of the disk before and after it. See Proposition 1.4.24, pages 53–54.

Riemann Sphere.

In many cases it is useful to extend a function $f : \mathbb{C} \to \mathbb{C}$ to infinity. To do this we define the Riemann sphere

$$\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$$

One way of doing this is **stereographic projection**. Regard $\mathbb{C}$ as the xy-plane in $\mathbb{R}^3$. Let $\mathbf{S}$ be the sphere given by

$$x^2 + y^2 + (z - \tfrac{1}{2})^2 = 1.$$

From the “North Pole,” that is (0,0,1), any downward ray with meet $\mathbf{S}$ at exactly one point, a, and $\mathbb{C}$ at exactly one point b. Define a map P that takes b to a for each $b \in \mathbb{C}$. Regard (0,0,1) as ∞. So, $\bar{\mathbb{C}}$ is just $\mathbf{S}$.

Suppose $f : \mathbb{C} \to \mathbb{C}$ has a the same limit $q \in \mathbb{C}$ for any infinite path $\gamma : [0, \infty) \to \mathbb{C}$ with $\lim_{t\to\infty} |\gamma(t)| = \infty$.

We can define a function $\bar{f} : \bar{\mathbb{C}} \to \mathbb{C}$ by $\bar{f}(a) = f(P(a))$.

If $q = \infty$, that is if $|f(\gamma(t))|$ goes to infinity as t does, we can define $\bar{f} : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ by $\bar{f}(a) = P^{-1}(f(P(a)))$ with $\bar{f}(\infty) = \infty$.

If f is undefined at some point z_o of $\mathbb{C}$ but the $\lim_{z\to z_o} |f(z)| = \infty$ we can define $\bar{f}(P^{-1}(z_o)) = \infty \in \bar{\mathbb{C}}$.

For example, we can now think of the map $f(z) = 1/z$ as being defined on all of $\bar{\mathbb{C}}$ with $f(0) = \infty$ and $f(\infty) = 0$.

One reason all this is helpful is that $\bar{\mathbb{C}}$ is compact, so continuous functions are uniformly continuous.