1.5 The Derivative.

<u>Def</u> $f'(z_0) = \lim_{z \to z_0} \frac{f(z_0) - f(z)}{z_0 - z}$, when the limit exists.

This looks just like the definition from Calc I. But it is different because the limit has to be the same along any path to z_0. And, its geometric meaning is different. This will have major implications.

<u>Vocab.</u> In Calc II you learned that a function $f: \mathbb{R} \to \mathbb{R}$ was <u>analytic</u> at x_0 if it was infinitely differentiable and the Taylor series converged to f on some open interval containing x_0. But your book says a function $f: \mathbb{C} \to \mathbb{C}$ is analytic if it is merely differentiable. Why? Because in Section 2.4 we will see that if f is diff. in the complex sense, then it must be infinitely diff. and in Section 3.2 that its Taylor series converges to f on at least an open disk. This is extremely surprising!

In complex analysis analytic functions are also called <u>holomorphic</u> functions. If $f: \mathbb{C} \to \mathbb{C}$ is diff. on all of $\mathbb{C}$ it is called an <u>entire</u> function.

The basic rules for computing derivatives are the same as in Calc I. (See Prop 1.5.3) Suppose f and g are diff. on an open subset A of $\mathbb{C}$. Then

$(af)' = af'$ where a in any complex constant.

$(f+g)' = f'+g'$.

$(fg)' = f'g + fg'$

$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$, provided $g \neq 0$.

$(z^n)' = n z^{n-1}$ for any integer n.

Ex $\left(\frac{3z^3+z-7}{z^2+1}\right)' = \frac{(9z^2+1)(z^2+1)-(3z^3+z-7)(2z)}{(z^2+1)^2}$, for $z \neq \pm i$.

Fact If $f'(z_0)$ exists, then f is cont. at z_0

The proof is the same as in Calc I. See Prop 1.5.2.

Fact The chain rule still works $\frac{d}{dz} g(f(z)) = g'(f(z)) \cdot f'(z)$.

Fact Let $A \subset \mathbb{C}$ be open and connected. If $f'(z) = 0$ on A, then f is constant on A. See Prop 1.5.5.

Ex The function $f(z) = \bar{z}$ is continuous on $\mathbb{C}$ but is not differential any where in $\mathbb{C}$.

For the proof of contnity recall we need to show, for any $z_0 \in \mathbb{C}$,

$$\lim_{z \to z_0} f(z) = f(z_0).$$

(See pg 44.) Let $z_0 \in \mathbb{C}$. Let $\varepsilon > 0$. Let $\delta = \varepsilon$. Then

$$|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon \quad \text{since,}$$

$$|f(z) - f(z_0)| = |\bar{z} - \bar{z}_0| = |\overline{z - z_0}| = |z - z_0| < \delta = \varepsilon.$$

Suppose $\lim_{z \to z_0} \frac{\bar{z} - \bar{z}_0}{z - z_0}$ exists. Then the limit along any path to z_0 will be the same. Let $z = x + iy$, $z_0 = x_0 + iy_0$.

Fix $y = y_0$ and consider $\lim_{x \to x_0} \frac{(x - iy_0) - (x_0 - iy_0)}{(x + iy_0) - (x_0 + iy_0)}$.

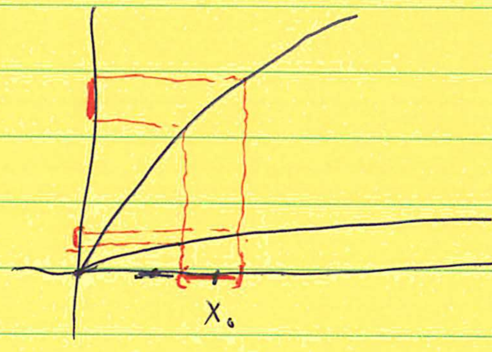
$$\text{It} = \lim_{x \to x_0} \frac{x - x_0}{x - x_0} = 1$$

Now fix $x = x_0$ and consider $\lim_{y \to y_0} \frac{(x_0 - iy) - (x_0 - iy_0)}{(x_0 + iy) - (x_0 + iy_0)}$

$= \lim_{y \to y_0} \frac{-iy + iy_0}{iy - iy_0} = -1$. Since $1 \neq -1$, $f'(z_0)$ does not exist.

What is the geometric meaning of $f'(z)$?

Recall: For $f: \mathbb{R} \to \mathbb{R}$, $f'(x_0)$ is the rate of change at x_0, or the slope of the tangent line to $y = f(x)$ at $x = x_0$. For $f'(x_0) \neq 0$ we can think of $|f'(x_0)|$ as the amount of stretching (or contracting) caused by f to a small interval containing x_0.

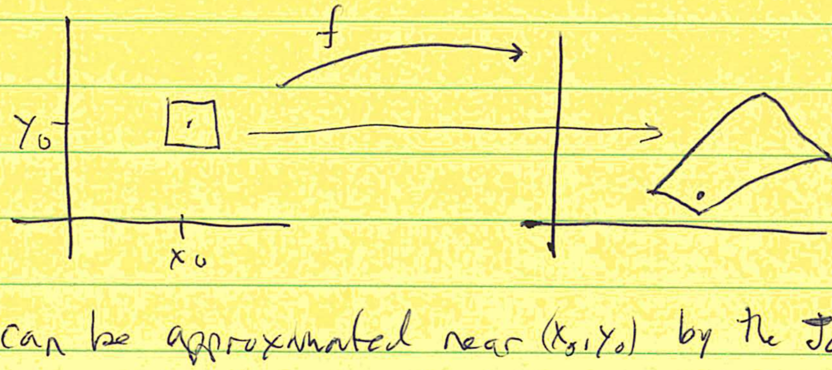


For $f: \mathbb{R}^2 \to \mathbb{R}$ the partial derivatives $\frac{\partial f}{\partial x}(x_0, y_0)$, $\frac{\partial f}{\partial y}(x_0, y_0)$ tell us the rate of change in the x and y directions resp.

For $f: \mathbb{R}^2 \to \mathbb{R}^2$ the absolute value of the determinant of the Jacobian matrix of partial derivatives

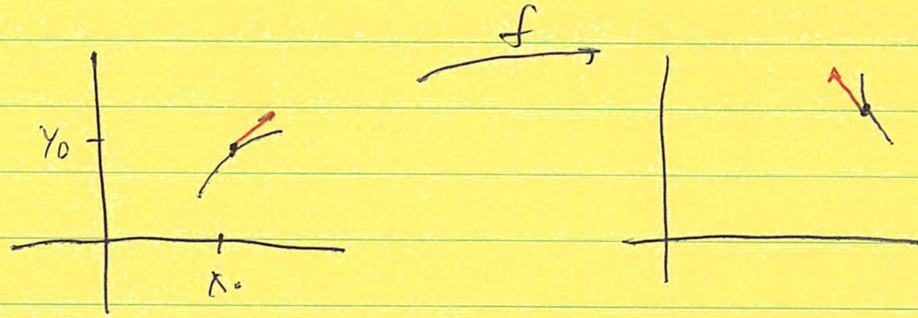
$$\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

tells us how much the area is being stretch (or contracted)



f can be approximated near (x_0, y_0) by the Jacobian.

Now for $f: \mathbb{C} \to \mathbb{C}$, $f'(z_0)$ is a complex number. Write it in polar form $re^{i\theta} = f'(z_0)$. The r tells much a patch around z_0 is stretched or contracted. θ tells points are rotated.

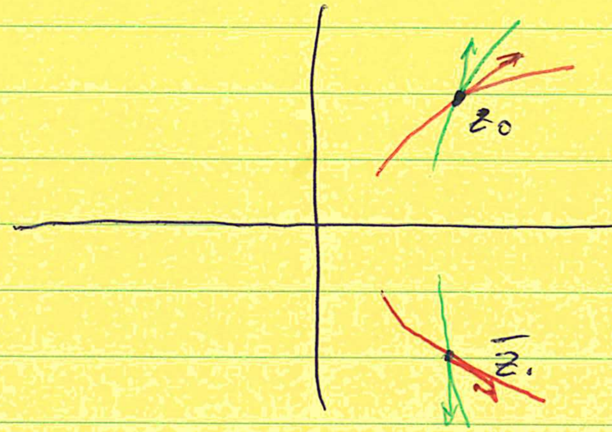


$z_0 = x_0 + iy_0$

The tangent vector to any smooth curve ~~at z_0~~ at z_0 is rotated θ.

<u>Note:</u> This fails if $f'(z_0) = 0$. But if $f'(z_0) \neq 0$ the angle between to tangent vectors is preserved. We say such a function is <u>conformal</u>.

Now we can <u>see</u> why $f(z) = \bar{z}$ is not differenticable.



The red and green tangent vectors are rotated by different amounts.

Even though $f: \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x,y) = (x, -y)$ has all partial derivative defined. In fact it is linear!

Cauchy-Riemann Equations.

Thm 1.5.8 Let $A \subset \mathbb{C}$ be open and $f : A \to \mathbb{C}$.

Then $f'(z_0)$ exists if and only if (let $f = u + iv$)

① $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist at $z_0 = x_0 + iy_0$ and

are continuous and

② the Cauchy-Riemann Equations hold

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{(Know These!)}$$

If $f'(z)$ exists, then $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$.

Ex Let $f(z) = \bar{z}$. Then $\frac{\partial u}{\partial x} = 1$ $\frac{\partial v}{\partial y} = -1$, so the CRE's fail and f is not differentiable.

Ex Let $f(z) = \sin x + i \sin y$. Then $\frac{\partial u}{\partial x} = \cos x$ and $\frac{\partial v}{\partial y} = \cos y$.

But $\sin x \neq \sin y$ on any open set. So $f(z)$ is not differentiable.

Ex Suppose $f(x+iy) = x\sin y + i\,v(x,y)$. Can we find $v(x,y)$ s.t. f is diff. in the complex sense?

Sol.

$$\frac{\partial u}{\partial x} = \sin y. \quad \text{Thus } \frac{\partial v}{\partial y} = \sin y \Rightarrow v = -\cos y + C_1(x).$$

$$\frac{\partial u}{\partial y} = x\cos y. \quad \text{Thus } -\frac{\partial v}{\partial x} = x\cos y \Rightarrow v = -\frac{1}{2}x^2\cos y + C_2(y).$$

No such $v(x,y)$ exists.

Ex Suppose $f(x+iy) = e^x\cos y + i\,v(x,y)$. Can we find $v(x,y)$ s.t. f is diff. in the complex sense?

Sol.

$$\frac{\partial u}{\partial x} = e^x\cos y. \quad \frac{\partial v}{\partial y} = e^x\cos y \Rightarrow v = e^x\sin y + C_1(x)$$

$$\frac{\partial u}{\partial y} = -e^x\sin y. \quad \frac{\partial v}{\partial x} = e^x\sin y \Rightarrow v = e^x\sin y + C_2(y)$$

Let $v(x,y) = e^x\sin y$.

Then f is diff in the complex sense.

Proof of Thm 1.5.8

Part I Assume $f'(z_0)$ exists. Let $f(\overset{x+iy}{z}) = u(x,y) + iv(x,y)$.

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$ Since this limit exists,

by definition the limit along any path to z_0 exists and has the same value. We will compute this limit first fixing $y = y_0$ and letting $x \to x_0$, and then by fixing $x = x_0$ and letting $y \to y_0$.

$$f'(z_0) = \lim_{x \to x_0} \frac{f(x+iy_0) - f(x_0+iy_0)}{(x+iy_0) - (x_0+iy_0)}$$

$$= \lim_{x \to x_0} \frac{u(x,y_0) + iv(x,y_0) - u(x_0,y_0) - iv(x_0,y_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{u(x,y_0) - u(x_0,y_0)}{x - x_0} + i \lim_{x \to x_0} \frac{v(x,y_0) - v(x_0,y_0)}{x - x_0}$$

$$= \frac{\partial u}{\partial x}(x_0,y_0) + i \frac{\partial v}{\partial x}(x_0,y_0)$$

$$f'(z_0) = \lim_{y \to y_0} \frac{f(x_0+iy) - f(x_0+iy_0)}{(x_0+iy) - (x_0+iy_0)}$$

$$= \lim_{y \to y_0} \frac{u(x_0,y) + i v(x_0,y) - u(x_0,y_0) - i v(x_0,y_0)}{i(y-y_0)}$$

$$= \frac{1}{i} \lim_{y \to y_0} \frac{u(x_0,y) - u(x_0,y_0)}{y-y_0} + \frac{i}{i} \lim_{y \to y_0} \frac{v(x_0,y) - v(x_0,y_0)}{y-y_0}$$

$$= -i \frac{\partial u}{\partial y}(x_0,y_0) + \frac{\partial v}{\partial y}(x_0,y_0).$$

Thus, at $z_0 = x_0 + i y_0$ we have

$$f'(z_0) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

Hence $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$.

<u>Intermisson</u> Multiplication of a complex number $x+iy$ by a given complex number $a+ib$ is the "same" as multiplying a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ by the matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

$$(a+ib)(x+iy) = ax - by + i(bx+ay) \in \mathbb{C}$$

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax-by \\ bx+ay \end{bmatrix} \in \mathbb{R}^2$$

<u>Part II</u> Assume f is differentiable as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$. ~~Ea~~ at (x_0, y_0). Then $Df = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$

gives an approximation of f near (x_0, y_0). If $u_x = v_y$ and $u_y = -v_x$ then $Df = \begin{bmatrix} u_x & -v_x \\ v_x & u_x \end{bmatrix}$.

Thus, we can approximate $f : \mathbb{C} \to \mathbb{C}$ near $z_0 = x_0 + iy_0$ by multiplicatoon by tu complex number

$$u_x + i v_x = v_y - i u_y.$$

By defin.tion $f'(z_0)$ exists and equals this number. $\square$

Note: The C-R Equations can be written in polar coordinates as

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

See page 68.

Fact Let $f: A \to \mathbb{C}$ be diff. on some open set A and assume f^{-1} exists and is diff. on $f(A) \subset \mathbb{C}$. Then

$$\frac{d f^{-1}}{dw}(w_0) = \frac{1}{f'(f^{-1}(w_0))}$$

Pf Let $z = f^{-1}(w)$. Then $f(z) = w$.

$$\frac{d f(z)}{dw} = \frac{dw}{dw} = 1$$

$$\| $$

$$f'(z) \cdot \frac{dz}{dw} = 1$$

$$\| $$

$$\frac{d f^{-1}(w)}{dw} = \frac{1}{f'(z)} = \frac{1}{f'(f^{-1}(w))}.$$

Book's Thm 1.5.10 assumes less and proves f^{-1} exists and is diff. using a result from multivariable (real) calculus covered in MATH 450.

Examples are in next section.

Def A function $u: \mathbb{R}^2 \to \mathbb{R}$ is <u>harmonic</u> if

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

<u>Prop 1.5.12</u> Let $f: \mathbb{C} \to \mathbb{C}$ be $f(\cancel{x}+iy) = u(x,y) + i\,v(x,y)$. If f is analytic, then u and v are harmonic.

Pf. $u_x = v_y$ and $u_y = -v_x$.

Thus $u_{xx} = v_{yx}$ $\quad u_{yy} = -v_{xy}$

Thus $u_{xx} = -u_{yy} \Rightarrow u_{xx} + u_{yy} = 0.$

<u>Vocab.</u> In this setting we say u and v are harmonic conjugates. For example, we showed earlier that $e^x \cos y$ and $e^x \sin y$ are harmonic conjugates.

<u>Prop 1.5.13.</u> Level curves of harmonic conjugates meeting at right angles.

<u>Pf</u> $\nabla u \cdot \nabla v = \langle u_x, v_x \rangle \cdot \langle u_y, v_y \rangle = u_x u_y + v_x v_y$

$$= -u_x v_x + v_x u_x = 0. \qquad \square$$

See example on webpage.