1.6 **Differentiation of Elementary Functions.**

Fact: The derivative of e^z exists and equals e^z for all $z \in \mathbb{C}$

Note: If a function $f: \mathbb{C} \to \mathbb{C}$ is differentiable $\forall z \in \mathbb{C}$, then we say f is entire.

Pf: By definition $e^z = e^x(\cos y + i \sin y)$ where $z = x+iy$, $x, y \in \mathbb{R}$. Let $u(x,y) = e^x \cos y$, $v(x,y) = e^x \sin y$. Thus $e^z = u + iv$.
Now

$$\frac{\partial u}{\partial x} = e^x \cos y \quad \frac{\partial u}{\partial y} = -e^x \sin y$$

$$\frac{\partial v}{\partial x} = e^x \sin y \quad \frac{\partial v}{\partial y} = e^x \cos y.$$

Thus the Cauchy-Riemann equations hold

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Thus e^z is differentiable and

$$\frac{de^z}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x(\cos y + i \sin y) = e^z. \ \square$$

Fact: $\sin z$ and $\cos z$ are entire and

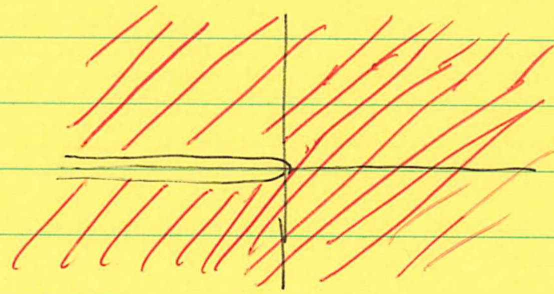
$$\frac{d \sin z}{dz} = \cos z \quad \text{and} \quad \frac{d \cos z}{dz} = -\sin z.$$

Pf: Use that $\sin z = \frac{e^{iz} - e^{-iz}}{2}$ and $\cos z = \frac{e^{iz} + e^{-iz}}{2}$.

Fact $\frac{d \log z}{dz} = \frac{1}{z}$, but, the domain of $\log z$

must be restricted. Recall $\log z = \ln|z| + i \arg z$. But $\arg z$ is not continuous. We usually use

$\mathbb{C} - \{ x+iy \mid x \le 0, y=0 \}$, that is $\mathbb{C}$ with 0 and the negative real axis removed, as the domain of $\log z$.



See Prop 1.6.2, pg 82, for proofs.

Power Functions For $n = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \ldots$ z^n is defined in the usual way and

$$\frac{d z^n}{dz} = n z^{n-1}.$$

For noninteger (real or complex) we define

$$z^n = e^{n \log z}$$

The usual power rule still work $\frac{d z^n}{dz} = n z^{n-1}$.

See Prop 1.6.4, pg 85, for proof.

Ex On what domain is $f(z) = \sqrt{z^2+1}$ differentiable and what is $f'(z)$?

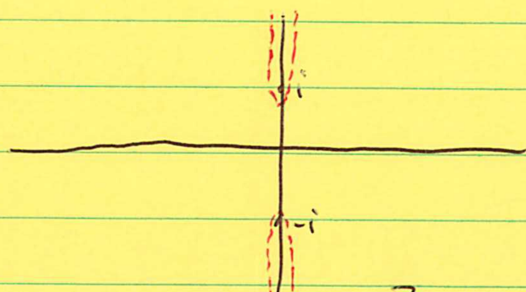
Sol The $\sqrt{\ }$ function is diff. on the same domain as $\log(\ )$, $\mathbb{C} - \{\text{neg } x\text{-axis and } 0\}$. Since z^2+1 is diff. everywhere $f(z)$ is diff. whenever z^2+1 is not on the neg real axis or 0.

$z^2+1 = 0$ for $z = \pm i$. Suppose z^2+1 is a neg. real number. Let $z = x+iy$. Now $z^2+1 = x^2-y^2+1 + 2ixy$. Thus, x or y is zero.

Suppose $y=0$. Then x^2+1 is neg., which is impossible. Thus, $x=0$. Hence $1-y^2<0$, i.e. $y \in (-\infty,-1)\cup(1,\infty)$.

Thus, the domain for which $f(z)$ is diff. is

$$\mathbb{C} - \{ yi \mid |y| \geq 1 \},$$



On this domain $f'(z) = \dfrac{z}{\sqrt{z^2+1}}$

Ex Let $f(z) = \sqrt{1+\sqrt{z}}$. For each $\sqrt{\ }$ use the branch that sends $z = re^{i\theta}$ to $\sqrt{r}\, e^{i\theta/2}$. Where is $f(z)$ diff.? Find $f'(z)$.

Sol We have to exclude the neg. real axis and 0 for $\sqrt{z}$ to be differentiable. Could it have that $1+\sqrt{z}$ lands ~~at~~ us on neg real axis or 0? No. This would give $\frac{\theta}{2} = \pi \Rightarrow \theta = 2\pi$ but then we'd use $\theta = 0$ for our branch and get a pos. real value. So, our domain is $\mathbb{C} - \{$neg real axis and $0\}$.

$$f'(z) = \frac{1}{2(1+\sqrt{z})} \cdot \frac{1}{2\sqrt{z}}.$$