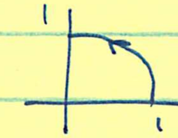
## 2.1 Contour Integrals in $\mathbb{C}$.

Recall line integrals from Calc III. Let $F:\mathbb{R}^2 \to \mathbb{R}^2$ be a vector field and let $C$ be a smooth curve given by $r:[a,b] \to \mathbb{R}^2$. Then

$$\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t)\, dt.$$




Ex Let $F = \langle x^2, -xy \rangle$, $C$ = quarter circle given by $r(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le \frac{\pi}{2}$. Compute $\int_C F \cdot dr$.

Sol.
$$r'(t) = \langle -\sin t, \cos t \rangle$$
$$F(r(t)) = \langle \cos^2 t, -\cos t \sin t \rangle$$
$$F \cdot r' = -s\,c^2 - c^2 s = -2c^2 s.$$

Thus,
$$\int_C F \cdot dr = \int_0^{\pi/2} -2\cos^2 t \sin t\, dt = \cancel{2\int_1 \cos^2 t\, dt}\ 2\int_1^0 u^2\, du = -\frac{2}{3}.$$

Let $u = \cancel{\sin}\cos t$.

Then $du = -\sin t\, dt$

[This is the work done pushing a particle along $C$ through the force field $F$.]

We are going to do something similar in the complex plane.

**Def** Let $[a,b] \subset \mathbb{R}$ and let $h:[a,b] \to \mathbb{C}$ be given by $h(t) = u(t) + iv(t)$ where $u$ and $v$ are real valued. Define

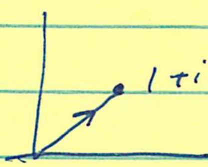
$$\int_a^b h(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

**Def** Let $f: \mathbb{C} \to \mathbb{C}$. Let $\gamma: [a,b] \to \mathbb{C}$ be a smooth curve in $\mathbb{C}$. Define

$$\int_\gamma f = \int_\gamma f(z)\,dz = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\,dt.$$

It is called a contour integral.

**Ex** $f(z) = z^3$, $\gamma(t) = t + ti$, $0 \le t \le 1$. Find $\int_\gamma f$.




**Sol.** $f(\gamma(t)) = (t+ti)^3$. $\gamma'(t) = 1+i$. Thus,

$$\int_\gamma f = \int_0^1 (t+ti)^3(1+i)\,dt = (1+i)^4 \int_0^1 t^3\,dt$$

$$= \frac{(1+i)^4}{4} = -1.$$

**Ex** Let $f(z) = 1$. Let $w, z \in \mathbb{C}$. Let $\gamma$ be the path from $z$ to $w$ given by $\gamma(t) = z + (w-z)t$, $0 \le t \le 1$.

$$\int_\gamma f = \int_0^1 1\,(w-z)\,dt = w - z.$$

(We will use this in the proof of Thm 2.1.9)

<u>Basic Facts</u>

ⓐ $\int_\gamma c_1 f + c_2 g = c_1 \int_\gamma f + c_2 \int_\gamma g.$

ⓑ $\int_{-\gamma} f = -\int_\gamma f$ ($-\gamma$ means go backwards. $-\gamma(t) = \gamma(b+(a-t$

ⓒ $\int_{\gamma_1+\gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f$

ⓓ $\int_a^b |\gamma'(t)|\,dt =$ arc length of $\gamma$ over $[a,b]$. $= \ell(\gamma)$

ⓔ $\left|\int_\gamma f\right| \leq \int_\gamma |f| \leq \max|f| \cdot \ell(\gamma).$ [Prop 2.1.6]

ⓕ Reparameterizations do not change $\int_\gamma f$.
[Prop. 2.1.5]

Proofs of a, b and c are easy. See textbook.

Pf of ⓓ Let $\gamma(t) = x(t) + i y(t)$. Then

$$\int_a^b |\gamma'(t)|\,dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$=$ arc length of $\gamma$ over $[a,b]$.

Proof of (e) [Prop 2.1.6] The proof is in two parts.

Let $g(t) = u(t) + iv(t)$. Then $\text{Re} \int_a^b g(t)\,dt = \int_a^b \text{Re}(g(t))\,dt$.

Let $\int_a^b g(t)\,dt = re^{i\theta}$. Then $r = e^{-i\theta} \int_a^b g(t)\,dt = \int_a^b e^{-i\theta} g(t)\,dt$.

Now $r = \text{Re}\, r = \text{Re} \int_a^b e^{-i\theta} g(t)\,dt = \int_a^b \text{Re}(e^{-i\theta} g(t))\,dt$.

Since $\text{Re}\, z \leq |z|$, $\text{Re}(e^{-i\theta} g(t)) \leq |e^{-i\theta} g(t)| = |g(t)|$

Thus, $\int_a^b \text{Re}(e^{-i\theta} g(t))\,dt \leq \int_a^b |g(t)|\,dt$

Hence, $\left| \int_a^b g(t)\,dt \right| = |re^{i\theta}| = r \leq \int_a^b |g(t)|\,dt$.

Now, for the second part of the proof we apply this to contour integrals.

$$\left| \int_\gamma f \right| = \left| \int_a^b f(\gamma(t))\, \gamma'(t)\,dt \right|$$

$$\leq \int_a^b | f(\gamma(t))\, \gamma'(t) |\,dt = \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt$$

$$\leq M \int_a^b |\gamma'(t)|\,dt = M\,\ell(\gamma). \qquad \square$$

Ex (#15 in 2.1) Prove that $\left| \int_\gamma \frac{\sin z}{z^2}\, dz \right| \le 2\pi e$

where $\gamma$ is the unit circle.

Sol. Let $\gamma = e^{it}$, $0 \le t \le 2\pi$. Then $\gamma'(t) = ie^{it}$.

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{ie^{it}} - e^{-ie^{it}}}{2i}$$

$$z^2 = e^{2it}$$

$$\left| e^{ie^{it}} \right| = \left| e^{i(\cos t + i\sin t)} \right| = \left| e^{-\sin t} e^{i\cos t} \right| = e^{-\sin t} \le e.$$

(same for $|e^{-ie^{it}}|$.)

$$\left| \frac{\sin z}{z^2} \right| = \left| \frac{e^{ie^{it}} - e^{-ie^{it}}}{2ie^{2it}} \right| = \frac{|e^{ie^{it}} - e^{-ie^{it}}|}{2} \le \frac{|e^{ie^{it}}| + |e^{-ie^{it}}|}{2}$$

$$\le \frac{e+e}{2} = e.$$

Arc length of $\gamma$ is $2\pi$.

$$\left| \int_\gamma \frac{\sin z}{z^2}\, dz \right| \le \int_0^{2\pi} \left| \frac{\sin e^{it}}{e^{2it}} \right| \cdot |\gamma'(t)|\, dt$$

$$\le \max\left( \frac{\sin e^{it}}{e^{2it}} \right) \cdot \ell(\gamma) \le e \cdot 2\pi. \qquad \square$$

Proof of (f)

**Def** Let $\gamma : [a,b] \to \mathbb{C}$ be a smooth curve.
Another smooth curve $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \to \mathbb{C}$ is called a reparameterization of $\gamma$ if $\exists$ smooth function

$$\alpha : [a,b] \to [\tilde{a}, \tilde{b}]$$

with $\alpha(a) = \tilde{a}$, $\alpha(b) = \tilde{b}$, $\alpha'(t) > 0$, and $\gamma(t) = \tilde{\gamma}(\alpha(t))$.

**Prop 2.1.5** $\int_\gamma f = \int_{\tilde{\gamma}} f$, where $f$ is cont. and defined on an open set containing the image of $\gamma$ (= image of $\tilde{\gamma}$).

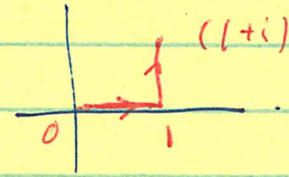
**Pf** $\gamma'(t) = \dfrac{d\gamma(t)}{dt} = \dfrac{d\tilde{\gamma}(\alpha(t))}{dt} = \tilde{\gamma}'(\alpha(t))\,\alpha'(t)$.

$$\int_\gamma f = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt = \int_a^b f(\tilde{\gamma}(\alpha(t)))\,\tilde{\gamma}'(\alpha(t))\,\alpha'(t)\,dt = *$$

Let $u = \alpha(t)$. Then $du = \alpha'(t)\,dt$. Hence

$$* = \int_{\tilde{a}}^{\tilde{b}} f(\tilde{\gamma}(u))\,\tilde{\gamma}'(u)\,du = \int_{\tilde{\gamma}} f. \qquad \square$$

We will do a few more examples, and then study path independence.

Ex Let $f(z) = z^3$. $\gamma$ is this path 

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Find $\int_\gamma f$.

Sol. Parameterize $\gamma$ by $\gamma(t) = \begin{cases} \gamma_1(t) = t & 0 \le t \le 1 \\ \gamma_2(t) = 1 + (t-1)i & 1 \le t \le 2 \end{cases}$

Use $\int_\gamma f = \int_{\gamma_1} f + \int_{\gamma_2} f$.

$$\int_{\gamma_1} f = \int_0^1 f(\gamma_1(t))\gamma_1'(t)\,dt = \int_0^1 (t)^3 \cdot 1\,dt = \frac{1}{4}.$$

$$\int_{\gamma_2} f = \int_1^2 f(\gamma_2(t))\gamma_2'(t)\,dt = \int_1^2 (1+(t-1)i)^3\, i\,dt$$

Let $s = t-1$. Then $ds = dt$.

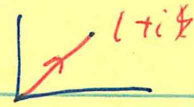
$$= \int_0^1 (1+si)^3 i\,ds = i\int_0^1 1 + 3si - 3s^2 - is^3\,ds$$

$$= i\left(s + \frac{3}{2}is^2 - \frac{3s^3}{3} - \frac{is^4}{4}\right)\Big|_0^1$$

$$= i\left(1 + \frac{3}{2}i - 1 - \frac{i}{4}\right) - i(0) = -\frac{3}{2} + \frac{1}{4} = -\frac{5}{4}.$$

Thus $\int_{\gamma_1} f + \int_{\gamma_2} f = \frac{1}{4} - \frac{5}{4} = -1$.

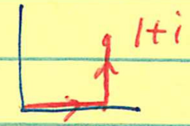
This is the same as when we took the direct path from 0 to $1+i$.

**Ex** Let $f(z) = \bar{z}$. $\gamma =$ 

. Find $\int_\gamma f$.

**Sol.** Let $\gamma(t) = (1+i)t$. $\quad 0 \le t \le 1$. $\gamma'(t) = 1+i$. $f(\gamma(t)) = (1-i)t$.

$$\int_\gamma f = \int_0^1 (1-i)\,t\,(1+i)\,dt = 2\int_0^1 t\,dt = \frac{2}{2} = 1.$$

**Ex** Let $f(z) = \bar{z}$ and $\gamma =$ 

. Find $\int_\gamma f$.

**Sol** Let $\gamma_1, \gamma_2$ be as in example before the last one.

$$\int_{\gamma_1} f = \int_0^1 t \cdot 1\,dt = \frac{1}{2}$$

$$\int_{\gamma_2} f = \int_1^2 (1-(t-1)i)\,i\,dt = \int_0^1 (1-si)\,i\,ds$$

$s = t-1$

$ds = dt$

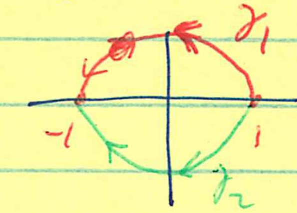
$$= i\left(s - \frac{s^2 i}{2}\right)\Big|_0^1 = i\left(1 - \frac{i}{2}\right) - i(0) = \frac{1}{2} + i.$$

Thus $\int_\gamma f = \frac{1}{2} + \frac{1}{2} + i = 1+i$. Not the same.

Sometimes path matters!

<u>Ex</u> (Example 2 in textbook, pg 104)

Let $f(z) = \frac{1}{z}$. We will find $\int_\gamma f$ for two paths from $+1$ to $-1$.




$\gamma_1(t) = e^{it}$, $0 \le t \le \pi$.

$\gamma_2(t) = e^{-it}$, $0 \le t \le \pi$.

$$\int_{\gamma_1} f = \int_0^\pi \frac{1}{e^{it}}(ie^{it})\,dt = i\int_0^\pi dt = i\pi.$$

$$\int_{\gamma_2} f = \int_0^\pi \frac{1}{e^{-it}}(-i\,e^{-it})\,dt = -i\int_0^\pi dt = -i\pi.$$

So, the path mattered.

Recall in Calc III path independence happened when the vector field was smooth and conservative.

$F(x,y) = \langle M(x,y), N(x,y) \rangle$ is conservative when

$F = \nabla f$, for some $f: \mathbb{R}^2 \to \mathbb{R}$. This

happens iff $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

**Thm 2.1.7** Let $\gamma : [a,b] \to \mathbb{C}$ be smooth. Let $F$ be diff. on an open set $G$ that contains $\gamma([a,b])$. (Assume $F'$ is cont.) Then,

$$\int_\gamma F'(z)\,dz = F(\gamma(b)) - F(\gamma(a)).$$

If $\gamma(a) = \gamma(b)$, then $\int_\gamma F'(z)\,dz = 0$.

**Pf** Let $g(t) = F(\gamma(t))$. Let $u(t) + iv(t) = g(t)$. Then

$$F'(\gamma(t))\gamma'(t) = g'(t) = u'(t) + iv'(t).$$

Thus, $\displaystyle\int_\gamma F'(z)\,dz = \int_a^b u'(t) + iv'(t)\,dt = [u(b) + iv(b)] - [u(a) + iv(a)]$

$$= g(b) - g(a) = F(\gamma(b)) - F(\gamma(a)). \qquad \square$$

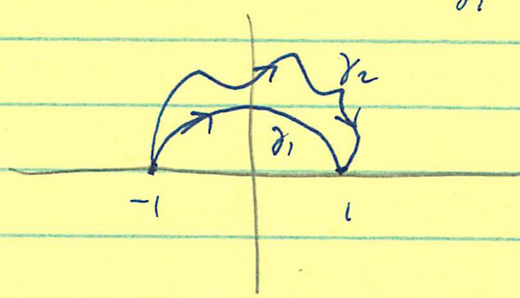
Works if $\gamma$ is piecewise smooth.

**Ex** Let $f(z) = z^3$. Let $\gamma$ be any path from $0$ to $1+i$. Then

$$\int_\gamma f(z)\,dz = \frac{z^4}{4}\Big|_0^{1+i} = \frac{(1+i)^4}{4} = -1.$$

**Ex** Since this fails for $f(z) = \bar{z}$, there is no function $F(z)$ s.t. $F'(z) = \bar{z}$.

What about the examples with $f(z) = \frac{1}{z}$ and the two paths from $-1$ to $1$? Let $F(z) = \log z$. Then $F'(z) = \frac{1}{z}$. But, the problem is you cannot use the same branch of the $\log z$ for both paths. However, the next theorem shows that for any two paths $\gamma_1, \gamma_2$ from $-1$ to $1$ that stay in the upper half plane, not going through $0$, will have the same integral: $\int_{\gamma_1} \frac{1}{z}\,dz = \int_{\gamma_2} \frac{1}{z}\,dz$




See extra example on the website where I compute $\int_\gamma \frac{1}{z}\,dz$ for a parabolic arc from $-1$ to $1$.

**Thm 2.1.9** Let $f$ be cont. in an open connected subset $G \subset \mathbb{C}$. The following are equivalent.

(i) Path independence: If $\gamma_1, \gamma_2$ are paths in $G$ from $z_0$ to $z_1$, then $\int_{\gamma_1} f = \int_{\gamma_2} f$.

(ii) If $\gamma$ is a closed loop in $G$, $\oint_\gamma f = 0$.

(iii) $\exists$ a function $F(z)$ s.t. $F'(z) = f(z)$ $\forall z \in G$.

Pf We already have (iii) $\Rightarrow$ (i) by 2.1.7. (i) $\Leftrightarrow$ (ii) is easy. We only need to show (i) $\Rightarrow$ (iii).

Let $z_0 \in G$. Let $z \in G$. Let $\gamma$ be a path from $z_0$ to $z$ in $G$. Think of $z_0$ as fixed and $z$ as variable. Define

$$F(z) = \int_\gamma f.$$

Since we are assuming (i) holds $F(z)$ depends only on $z$ (and $z_0$, but not $\gamma$). Consider the limit

$$\lim_{w \to z} \frac{F(w) - F(z)}{w - z}.$$

If we can show this limit exist and equals $f(z)$ we will have (iii).

Let $\varepsilon > 0$. We will find a $\delta > 0$ s.t. $|w - z| < \delta \Rightarrow$

$$\left| \frac{F(w) - F(z)}{w - z} - f(z) \right| < \delta.$$

Hence, by the definition of a limit

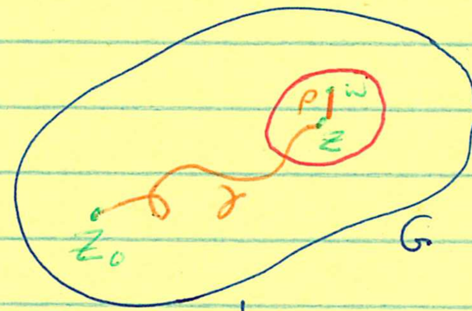
$$\lim_{w \to z} \frac{F(w) - F(z)}{w - z} = f(z).$$

Since $G$ is open, $\exists\, \delta_1 > 0$ s.t. $D(z, \delta_1) \subset G$.
Since $f$ is cont, $\exists\, \delta_2 > 0$ s.t. $w \in D(z, \delta_2) \Rightarrow f(w) \in D(f(z), \varepsilon)$.

Let $\delta = \min(\delta_1, \delta_2)$. Suppose $|w - z| < \delta$.

Let $p(t)$ be a straight path from $z$ to $w$.
Then $p(t)$ is in $D(z, \delta)$ and hence in $G$.

Now $F(w) - F(z) = \int_{\gamma + p} f - \int_\gamma f = \int_p f$.




Thus,

$$\left| \frac{F(w) - F(z)}{w - z} - f(z) \right| = \left| \frac{F(w) - F(z) - (w - z) f(z)}{w - z} \right|$$

$$= \frac{|F(w) - F(z) - (w-z) f(z)|}{|w - z|} = \frac{\left| \int_p f(q)\, dq - f(z) \int_p 1\, dq \right|}{|w - z|}$$

$\int_p 1\, dq = w - z$

$$= \frac{\left| \int_p f(q) - f(z)\, dq \right|}{|w - z|} \le \frac{\max |f(q) - f(z)| \cdot \text{length}(p)}{|w - z|}$$

$$< \frac{\varepsilon \cdot |w - z|}{|w - z|} = \varepsilon,$$

Since $f(q)$ is always inside $D(f(z), \varepsilon)$.

$\square$