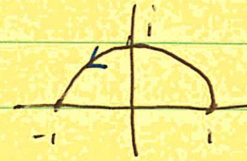
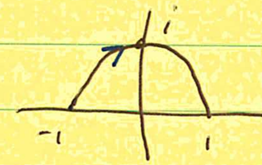
Ex Example 2, pg 109, showed that $\int_{\gamma_1} \frac{1}{z}\,dz = +i\pi$, where

γ_1 is the upper half of the unit circle, $\gamma_1(t) = e^{it}$, for $0 \le t \le \pi$.



Let γ_3 be the parabolic arc: 

.

Will the integral $\int_{\gamma_3} \frac{1}{z}\,dz$ "still" be $-i\pi$? (I switch the direction.)

Step 1 Find a parameterization of γ_3. Let $\gamma_3(t) = t + (1-t^2)i$, for $-1 \le t \le 1$.

Step 2 Set up the integral. $\gamma_3'(t) = 1 - 2ti$. Thus

$$\int_{\gamma_3} \frac{1}{z}\,dz = \int_{-1}^{1} \frac{1}{\gamma_3(t)}\,\gamma_3'(t)\,dt = \int_{-1}^{1} \frac{1-2ti}{t+(1-t^2)i}\,dt$$

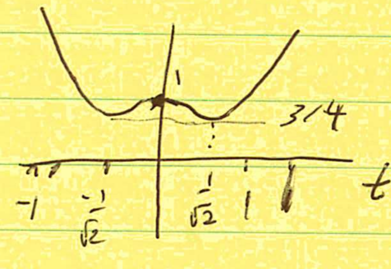
$$= \int_{-1}^{1} \frac{1-2ti}{t+(1-t^2)i} \cdot \frac{t-(1-t^2)i}{t-(1-t^2)i}\,dt$$

$$= \int_{-1}^{1} \frac{[t + 2t(1-t^2)] + [-2t^2 - (1-t^2)]i}{t^2 + (1-t^2)^2}\,dt$$

$$= \int_{-1}^{1} \frac{[-2t^3+3t] - [t^2+1]i}{t^4 - t^2 + 1}\,dt$$

$$= \int_{-1}^{1} \frac{-2t^3+3t}{t^4-t^2+1}\,dt - i\int_{-1}^{1} \frac{t^2+1}{t^4-t^2+1}\,dt$$

Step 3 Evaluate the real part.



The graph of t^4-t^2+1 is
The mins occur at $t=\pm\frac{1}{\sqrt{2}}$
and are $\frac{3}{4}$. So this function
is never zero and hence we do not need to break up
the integrals.

Now $-2t^3+3t$ is odd and t^4-t^2+1 is even. Thus,
$\frac{-3t^3+3t}{t^4-t^2+1}$ is odd. Hence $\int_{-1}^{1}\frac{-3t^3+3}{t^4-t^2+1}\,dt=0$.

Step 4 Evaluate the imaginary part. This will be harder.
We will apply the method of partial fraction to $\frac{t^2+1}{t^4-t^2+1}$.
To do this we must fact t^4-t^2+1 over the
reals. We know it roots are complex so we need to
factor it into two real quadratic functions. (There is
a thm that this can always be done.)

Step 4a Factoring t^4-t^2+1. We find the complex roots.
setting it to zero, the quadratic formula gives

$$t^2=\frac{1\pm\sqrt{1-4}}{2}=\frac{1}{2}\pm i\frac{\sqrt{3}}{2}.\quad \text{Thus } t=\pm\sqrt{\frac{1}{2}\pm\frac{i\sqrt{3}}{2}}.$$

$$\text{Thus } t^4-t^2+1=\underset{(a)}{\left(t+\sqrt{\frac{1}{2}+\frac{i\sqrt{3}}{2}}\right)}\underset{(b)}{\left(t-\sqrt{\frac{1}{2}+\frac{i\sqrt{3}}{2}}\right)}\underset{(c)}{\left(t+\sqrt{\frac{1}{2}-\frac{i\sqrt{3}}{2}}\right)}\underset{(d)}{\left(t-\sqrt{\frac{1}{2}-\frac{i\sqrt{3}}{2}}\right)}$$

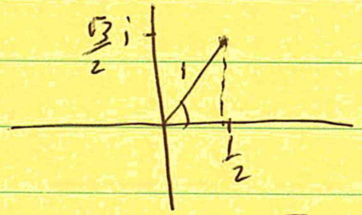
$$a \times c = \left(t+\sqrt{\frac{1}{2}+\frac{i\sqrt{3}}{2}}\right)\left(t+\sqrt{\frac{1}{2}-\frac{i\sqrt{3}}{2}}\right)$$

$$= t^2 + \left(\sqrt{\frac{1}{2}-\frac{i\sqrt{3}}{2}}+\sqrt{\frac{1}{2}+\frac{i\sqrt{3}}{2}}\right)t + \sqrt{\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)}$$

The last term is $\sqrt{\frac{1}{4}+\frac{3}{4}} = \sqrt{1} = 1$.

The coeff of must real (why?). We can simplify it.

$\sqrt{\frac{1}{2}+i\frac{\sqrt{3}}{2}} = \sqrt{e^{i\theta}} = e^{i\frac{\theta}{2}}$. We can get θ from a picture.



$$\cos\theta = \frac{1}{2}, \quad \sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3}.$$

Thus $\sqrt{\frac{1}{2}+i\frac{\sqrt{3}}{2}} = e^{i\frac{\pi}{6}} = \frac{\sqrt{3}}{2}+\frac{i}{2}$.

Likewise, $\sqrt{\frac{1}{2}-i\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{2}-\frac{i}{2}$. Hence their sum is $\sqrt{3}$.

Thus $t^2+\sqrt{3}t+1$ is a factor of t^4-t^2+1.

The other factor is $t^2-\sqrt{3}t+1$, as you can check.

$$(t^2+\sqrt{3}t+1)(t^2-\sqrt{3}t+1) = t^4-\sqrt{3}t^3+\sqrt{3}t^3-3t^2+t^2+t^2+1$$

$$= t^4-t^2+1.$$

Step 4b Find partial fraction expansion.

$$\frac{t^2+1}{(t^2+\sqrt{3}t+1)(t^2-\sqrt{3}t+1)} = \frac{At+B}{t^2+\sqrt{3}t+1} + \frac{Ct+D}{t^2-\sqrt{3}t+1}$$

We cross multiply to get

$$(A+C)t^3+(B+D-\sqrt{3}A+\sqrt{3}C)t^2+(A+C-\sqrt{3}B+\sqrt{3}D)t+(B+D) = t^2+1.$$

Thus,
$$\begin{aligned} A+C &= 0 && \Rightarrow A=-C \\ B+D-\sqrt{3}A+\sqrt{3}C &= 1 && \\ A+C-\sqrt{3}B+\sqrt{3}D &= 0 && \Rightarrow B=D \\ B+D &= 1. && \Rightarrow B=D=\tfrac{1}{2}. \end{aligned}$$

Then $-\sqrt{3}A-\sqrt{3}A = 0 \Rightarrow A=0 \Rightarrow C=0.$

Thus, our integral now looks like

$$\int_{-1}^{1} \frac{\frac{1}{2}}{t^2+\sqrt{3}t+1} + \frac{\frac{1}{2}}{t^2-\sqrt{3}t+1}\,dt$$

Step 4c For each we will need to complete the square.
Recall
$$t^2+bt+c = (t+P)^2+Q = t^2+2Pt+P^2+Q$$

$$\Rightarrow b=2P,\ c=P^2+Q \Rightarrow P=\frac{b}{2},\ Q=c-\frac{b^2}{4}.$$

Now then,

$$t^2+\sqrt{3}t+1 = t^2+\sqrt{3}t+\frac{3}{4}+\frac{1}{4} = \left(t+\frac{\sqrt{3}}{2}\right)^2+\frac{1}{4}.$$

$$t^2-\sqrt{3}t+1 = \qquad = \left(t-\frac{\sqrt{3}}{2}\right)^2+\frac{1}{4}.$$

<u>Step 4d</u> Do the integrals.

$$\frac{1}{2}\int_{-1}^{1}\frac{1}{\left(t+\frac{\sqrt{3}}{2}\right)^2+\frac{1}{4}}\,dt \qquad \text{Let } u=t+\frac{\sqrt{3}}{2}.\ du=dt$$

$$=\frac{1}{2}\int\frac{1}{u^2+\frac{1}{4}}\,dt = \frac{1}{2}\int\frac{4}{4}\,\frac{1}{u^2+\frac{1}{4}}\,du = 2\int\frac{1}{(2u)^2+1}\,du$$

Let $w=2u$. $du=\frac{1}{2}dw$. $\quad = \int\frac{1}{w^2+1}\,dw = \arctan(w)\Big|$

$$=\arctan\left(2\left(t+\frac{\sqrt{3}}{2}\right)\right)\Big|_{-1}^{1} = \arctan(2t+\sqrt{3})\Big|_{-1}^{1} =$$

$$\arctan(2+\sqrt{3})-\arctan(-2+\sqrt{3})$$

Likewise, $\frac{1}{2}\int_{-1}^{1}\frac{1}{\left(t-\frac{\sqrt{3}}{2}\right)^2+\frac{1}{4}} = \arctan(2t-\sqrt{3})\Big|_{-1}^{1}$

$$=\arctan(2-\sqrt{3})-\arctan(-2-\sqrt{3})$$

Thus, the imaginary part of the intergal is

$$-\left(\arctan(2+\sqrt{3})-\arctan(-2+\sqrt{3})+\arctan(2-\sqrt{3})-\arctan(-2-\sqrt{3})\right).$$

Since the arctan is odd this is

$$-\left(2\arctan(2+\sqrt{3}) + 2\arctan(2-\sqrt{3})\right).$$

$\arctan(2+\sqrt{3}) = 75^\circ$ or $\frac{5\pi}{12}$ and $\arctan(2-\sqrt{3}) = 15^\circ$ or $\frac{\pi}{12}$.

Thus, imaginary part is $-2\left(\frac{5\pi}{12} + \frac{\pi}{12}\right) = -\pi$.

And so the original integral is $-i\pi$ after all!

If you don't have a calculator to find the arctan's, here is a trick. Let $\theta_1 = \arctan(2+\sqrt{3})$ and $\theta_2 = \arctan(2-\sqrt{3})$. Then

$$\tan\theta_1 = 2+\sqrt{3} \quad \text{and} \quad \tan\theta_2 = 2-\sqrt{3}.$$

Thus

$$\tan\theta_1 \tan\theta_2 = (2+\sqrt{3})(2-\sqrt{3}) = 4-3 = 1.$$

Hence

$$\sin\theta_1 \sin\theta_2 = \cos\theta_1 \cos\theta_2$$

$$\Rightarrow 0 = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$$

$$\Rightarrow 0 = \cos(\theta_1 + \theta_2).$$

Thus $\theta_1 + \theta_2 = \frac{\pi}{2}$ as the result is the same.