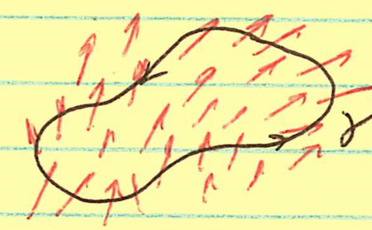
2.2

Cauchy's Theorem

We will need to use Green's Theorem.

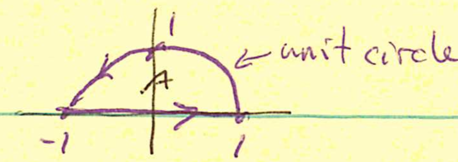
Let $F = \langle P(x,y), Q(x,y) \rangle$ be a vector field on $\mathbb{R}^2$. Assume the partial derivatives of P and Q exist and are continuous.

Let $\gamma : [a,b] \to \mathbb{R}^2$ be a piecewise smooth simple closed curve ($\gamma(a) = \gamma(b)$). Let A denote the region inside γ. ~~[illegible]~~ Assume $\gamma(t)$ is oriented ccw.



Then $\oint_\gamma F \cdot dr = \oint_\gamma P(x,y)\,dx + Q(x,y)\,dy$

$$= \iint_A \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dx\,dy$$

Ex Let $F = \langle \overset{P}{2x - y}, \overset{Q}{x} \rangle$. Let $\gamma =$ 

Check that $\oint_\gamma F \cdot dr = \iint_A \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dx\,dy$.

Sol. $\oint_\gamma F \cdot dr = \int_{\gamma_1} F \cdot dr + \int_{\gamma_2} F \cdot dr$ where

$\gamma_1 = (t, 0) \quad -1 \le t \le 1$ and $\gamma_2(t) = (\cos t, \sin t) \quad 0 \le t \le \pi$.

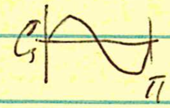
$$\oint_{\gamma_1} P(x,y)\,dx + Q(x,y)\,dy = \int_{-1}^{1} (2t - 0) \cdot 1 \, dt + t \cdot 0 \, dt$$

$$= 2\int_{-1}^{1} 2\,dt = 0$$

$$\oint_{\gamma_2} P\,dx + Q\,dy = \int_0^\pi (2\cos t - \sin t)(-\sin t)\,dt + \int_0^\pi \cos t \cos t \, dt$$

$$= \int_0^\pi -2\cos t \sin t + \sin^2 t + \cos^2 t \, dt$$

$$= \int_0^\pi -\sin 2t + 1 \, dt = \pi.$$



Thus, $\oint_\gamma F \cdot dr = \pi$.

Next $\iint_A \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dx\,dy = \iint_A 1 - (-1) \, dx\,dy$

$= 2 \times$ area of $\frac{1}{2}$ unit disk $= 2 \cdot \frac{1}{2} \cdot \pi \cdot 1^2 = \pi.$

And $\pi = \pi$! $\square$

<u>Thm 2.2.1</u> (Cauchy's Thm) Let $f: \mathbb{C} \to \mathbb{C}$ be differentiable with $f'(z)$ cont. on and inside a piecewise smooth simple closed curve given by $\gamma(t) = x(t) + iy(t) : [a,b] \to \mathbb{C}$ ccw. Then

$$\oint_\gamma f = 0.$$

<u>Pf</u> Let A be the region inside γ. Let $f = u + iv$.

$$\oint_\gamma f = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt = \int_a^b [u(\gamma(t)) + iv(\gamma(t))][x'(t) + iy'(t)]\, dt$$

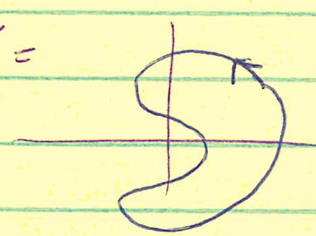
$$= \int_a^b u(\gamma(t))\, x'(t) - v(\gamma(t))\, y'(t)\, dt \; +$$

$$i \int_a^b v(\gamma(t))\, x'(t) + u(\gamma(t))\, \gamma'(t)\, dt$$

$$= \oint_\gamma u\, dx - v\, dy + i \oint_\gamma v\, dx + u\, dy$$

$$= \iint_A -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\, dx dy + i \iint_A \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\, dx dy$$

$= 0 + i0 = 0$ because of the Cauchy-Riemann Equations. $\square$

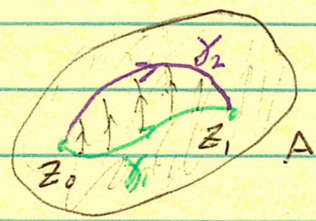
<u>Ex</u> $\int_\gamma z^3 + z^2 + \sin z\, dz \quad \gamma =$ 

$= 0$

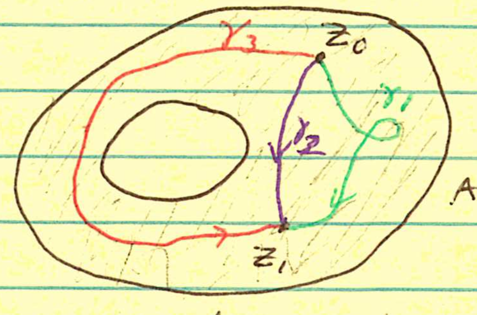
Formal definitions are in section

Homotopy and Simply Connected Regions (2.3)

Idea Let $A \subset \mathbb{C}$ be connected. Two curves from z_0 to z_1 in A are homotopic if you can be "gradually deformed" to the other within A.

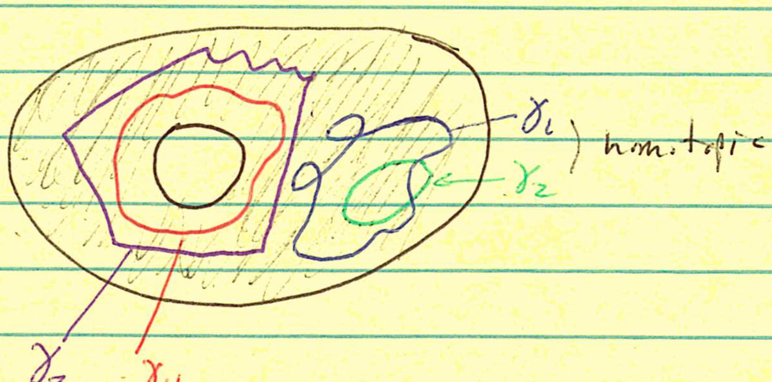


Two homotopic curves



γ_1 and γ_2 are homotopic A, but γ_3 is not homotopic to either.

Two closed curves in A are homotopic as closed curves if one can be "gradually deformed" to the other within A.



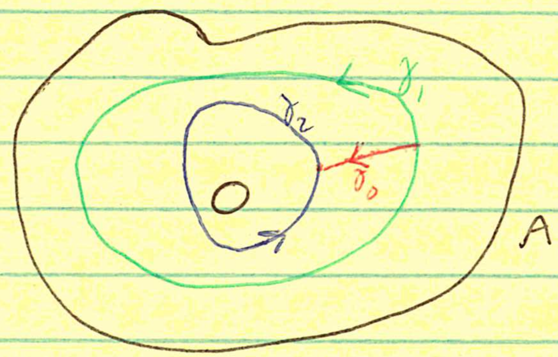
homotopic

If every closed curve in A is homotopic to a point (i.e. a map $\gamma(t) = w \ \forall t \in [a,b]$), then A is said simply connected.

Thm 2.2.2 Let $A \subset \mathbb{C}$ be open, connected (but not necessarily simply connected). Let $f: A \to \mathbb{C}$ be diff. on A. Let γ_1 and γ_2 be two simple closed curves in A. If they are homotopic as closed curves in A then

$$\int_{\gamma_1} f = \int_{\gamma_2} f.$$

Intuitive "Proof":



Let γ be $\gamma_1 + \gamma_0 - \gamma_2 - \gamma_0$. Then $\int_\gamma f = 0$.

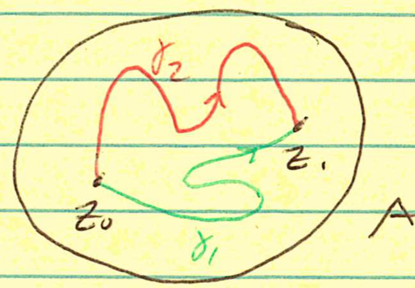
But $\int_\gamma f = \int_{\gamma_1} f + \int_{\gamma_0} f - \int_{\gamma_2} f - \int_{\gamma_0} f = \int_{\gamma_1} f - \int_{\gamma_2} f$.

Thus, $\int_{\gamma_1} f = \int_{\gamma_2} f$.

Prop 2.2.4 Let $A \subset \mathbb{C}$ be open, connected and simply connected. Let $f: A \to \mathbb{C}$ be diff. on A. Let $z_0, z_1 \in A$. Let γ_1 and γ_2 be two (simple) curves from z_0 to z_1, in A. Then

$$\int_{\gamma_1} f = \int_{\gamma_2} f.$$

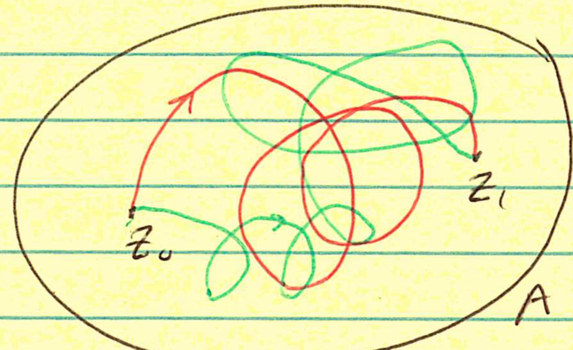
(simple) **Pf** Let $\gamma = \gamma_1 - \gamma_2$. It is a closed curve (simple for now). Thus Thm 2.2.1



$$\int_\gamma f = 0$$

Since $\int_\gamma f = \int_{\gamma_1} f - \int_{\gamma_2} f$ we have $\int_{\gamma_1} f = \int_{\gamma_2} f$. $\square$

Note: Works for "non simple" curves.



Ex Let $K \geq 2$. Let $\gamma(t)$ be a simple closed curve in $\mathbb{C}$ that does not pass through the origin. Show

$$\int_\gamma \frac{1}{z^k}\, dz = 0.$$

Sol. If the origin is not in the region enclosed by γ then Cauchy's Thm gives $\int_\gamma z^{-k}\, dz = 0$, since z^{-k} is diff. inside γ.

If γ does contain 0 inside the region it encloses then we can replace γ by the unit circle. Then

$$\int_\gamma \frac{1}{z^k}\, dz = \int_0^{2\pi} \frac{1}{e^{ikt}}\, ie^{it}\, dt = i\int_0^{2\pi} e^{i(1-k)t}\, dt$$

$$= i\int_0^{2\pi} \cos((1-k)t) + i\sin((1-k)t)\, dt = 0$$

Ex Let $f(z) = \frac{1}{z^8} - \frac{5}{z^3} + \frac{2}{z} + 3 + z^2 - 7z^5$.

Let γ be a scc that encloses 0. Find $\int_\gamma f$.

Sol. Let γ' be the unit circle. Then $\int_\gamma f = \int_{\gamma'} f$.

And $\int_{\gamma'} f = \int_{\gamma'} \frac{2}{z}\, dz = 2(2\pi i) = 4\pi i$.

Ex (#11) Let γ be the circle of radius 3, center 0, ccw.
Compute

$$\int_\gamma \frac{2z^2 + 15z + 30}{z^3 - 10z^2 + 32z - 32}\, dz$$

$$\frac{2z^2 - 15z + 30}{(z-2)(z-4)^2} = \frac{2}{z-2} + \frac{1}{(z-4)^2}$$

$4\pi i$