Thm (2.3.2) Cauchy's Theorem on a Disk.

Let $D = D(z_0, \rho)$, $\rho > 0$, $z_0 \in \mathbb{C}$.
Suppose $f : D \to \mathbb{C}$ is diff. on D.
Then,

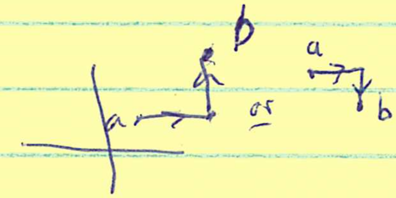
(i) f has an anti-derivative on D, $F'(z) = f(z)$, $\forall z \in D$.

(ii) If γ is a closed curve, piecewise smooth, (ie $\gamma'(t)$ is cont except for a finite number of kinks), then $\int_\gamma f = 0$.

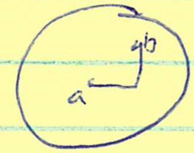
Strategy 0. From Thm 2.1.9 (i) $\Leftrightarrow$ (ii).

1. We have (ii) for rectangles. Use this to construct an anti-derivative in D.

Notation Let $a, b, \in \mathbb{C}$. Then $\langle\langle a, b \rangle\rangle$ = path



Notice, $\forall b \in D(a, r)$, $\langle\langle a, b \rangle\rangle \subset D(a, r)$.



Definition Let $F(z) = \int_{\langle\langle z_0, z \rangle\rangle} f(w)\, dw.$

We claim $F'(z) = f(z)$, $\forall z \in D$.

To prove this we study $\lim_{w \to z} \frac{F(w) - F(z)}{w - z}$, hoping it is $f(z)$.

Fix $z \in D$ and let $\varepsilon > 0$.

$\exists \delta_1 > 0$ s.t. $D(z, \delta_1) \subset D$ (since D is open).

$\exists \delta_2 > 0$ s.t. $q \in D(z, \delta_2) \Rightarrow |f(z) - f(q)| < \varepsilon$,
since f is cont.

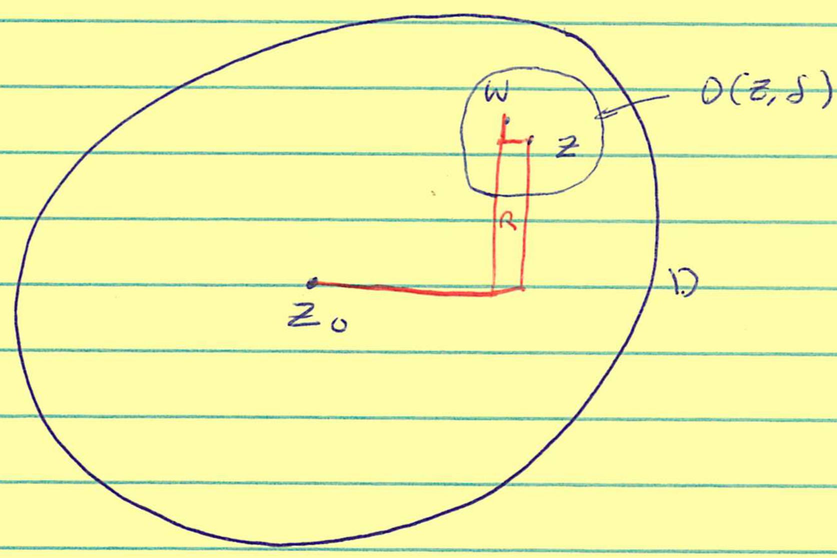
Let $\delta = \min\{\delta_1, \delta_2\}$.

For $w \in D(z, \delta)$, $\langle\langle z, w \rangle\rangle \subset D(z, \delta) \subset D$.

Also $\langle\langle z_0, z \rangle\rangle$ and $\langle\langle z_0, w \rangle\rangle$ are in D.

These paths determine a rectangle $\boxed{R}$ in D as follows:

Its corners are z, the corner of $\langle\langle z_0, w \rangle\rangle$,
the corner of $\langle\langle z_0, w \rangle\rangle$ and the corner of $\langle\langle z, w \rangle\rangle$.



Orient R ccw. Then $\int_R f = 0$.

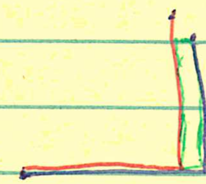
Now $F(z) = \int_{\langle\langle z_0, z\rangle\rangle} f(q)\,dq$ and $F(w) = \int_{\langle\langle z_0, w\rangle\rangle} f(q)\,dq$.

But,

$$\underbrace{\int_{\langle\langle z_0, w\rangle\rangle} f(q)\,dq}_{= F(w)} = \underbrace{\int_{\langle\langle z_0, z\rangle\rangle} f(q)\,dq}_{= F(z)} \pm \underbrace{\int_R f(q)\,dq}_{= 0} + \underbrace{\int_{\langle\langle z, w\rangle\rangle} f(q)\,dq}_{\text{short!}}$$

Thus, $F(w) - F(z) = \int_{\langle\langle z, w\rangle\rangle} f(q)\,dq$.

short!

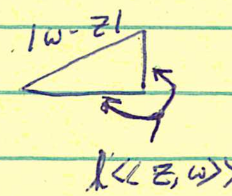


Now, $$\left|\frac{F(w)-F(z)}{w-z} - f(z)\right| = \frac{1}{|w-z|}\left|F(w)-F(z) - f(z)(w-z)\right|$$

$$= \frac{1}{|w-z|}\left|\int_{\langle\langle z, w\rangle\rangle} f(q)\,dq - f(z)\int_{\langle\langle z, w\rangle\rangle} 1\,dq\right|$$

$$= \frac{1}{|w-z|}\left|\int_{\langle\langle z, w\rangle\rangle} f(q) - f(z)\,dq\right| < \frac{1}{|w-z|}\,\varepsilon \cdot \ell(\langle\langle z, w\rangle\rangle)$$

$$\leq \frac{1}{|w-z|}\,\varepsilon \cdot 2|w-z| = 2\varepsilon.$$



Thus, $\lim_{w \to z} \frac{F(w)-F(z)}{w-z} = f(z)$ as claimed. ∎