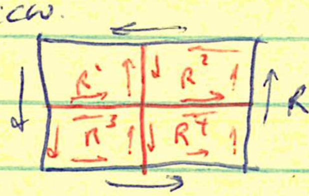
Thm (2.3.1) Cauchy's Thm on a Retangle.

Suppose R is a rectangle with sides parallel to the axes in $\mathbb{C}$. Let f be differentiable on an open set G containing R and its interior. Then

$$\oint_R f = 0.$$

Pf Let P = total lengths of the edges of R, ie its perimeter.
Let Δ = length of the diagonal of R.

Divide R into four congruent smaller rectangles, R^1, R^2, R^3, R^4.
Orient them ccw.



Then

$$\int_R f = \int_{R^1} f + \int_{R^2} f + \int_{R^3} f + \int_{R^4} f$$ (these a contour integrals around the perimeters.)

Now,

$$\left|\int_R f\right| \le \left|\int_{R^1} f\right| + \left|\int_{R^2} f\right| + \left|\int_{R^3} f\right| + \left|\int_{R^4} f\right|.$$

Thus, for at least one $k \in \{1,2,3,4\}$, $\left|\int_{R^k} f\right| \ge \frac{1}{4}\left|\int_R f\right|$.
(If not you can drive a contradiction!)
Call this R^k, R_1. Let $P_1 = \frac{1}{2}P$ and $\Delta_1 = \frac{1}{2}\Delta$.

Now subdivide R_1 and repeat to get R_2 with
$\left|\int_{R_2} f\right| \ge \frac{1}{4}\left|\int_{R_1} f\right| \ge \frac{1}{4^2}\left|\int_R f\right|$, $P_2 = \frac{1}{4}P$, $\Delta_2 = \frac{1}{4}\Delta$.

Keep doing this. Get $R_1, R_2, R_3, \ldots, R_n, \ldots$ such that

$$\text{(i)} \left|\int_{R_n} f\right| \geq \frac{1}{4^n}\left|\int_R f\right| \quad \text{(ii)} \ P_n = \frac{P}{2^n}, \quad \text{(iii)} \ \Delta_n = \frac{\Delta}{2^n}.$$

Let z_n be the upper left corner of R_n, for $n=1,2,3,\ldots$.
Then the sequence (z_n) is Cauchy since for $m>n$

$$|z_n - z_m| \leq \frac{\Delta}{2^n}, \text{ which goes to zero.}$$

Thus (z_n) converges. Let $z_n \to w_0$.

Remember, we are trying to show $\int_R f = 0$. So far we have

$$\left|\int_R f\right| \leq 4^n \left|\int_{R_n} f\right|, \text{ but } R_n \text{ is tiny.}$$

Now we use the differentiability of f at w_0.
Let $\varepsilon > 0$. $\exists \delta > 0$ s.t.

$$|z-w_0| < \delta \Rightarrow \left|\frac{f(z)-f(w_0)}{z-w_0} - f'(w_0)\right| < \varepsilon.$$

Choose n large enough that $\frac{\Delta}{2^n} < \delta$. Then $\forall z$ inside ~~on~~ or on R_n we have

$$|z-w_0| \leq \frac{\Delta}{2^n} < \delta.$$

Thus,

$$\left|f(z) - f(w_0) - (z-w_0)f'(w_0)\right| < \varepsilon|z-w_0| < \varepsilon\frac{\Delta}{2^n} \quad *$$

By the Path Independence Thm (2.1.9)

$$\int_{R_n} 1 \, dz = 0 \quad \text{and} \quad \int_{R_n} (z - w_0) \, dz = 0$$

Since 1 and $(z - w_0)$ have obvious anti-derivatives.

Now we get tricky.

$$\left| \int_R f \right| \le 4^n \left| \int_{R_n} f \right| = 4^n \left| \int_{R_n} f - f(w_0) \overbrace{\int_{R_n} 1}^{0} - f'(w_0) \overbrace{\int_{R_n} z - w_0}^{0} \right|$$

$$= 4^n \left| \int_{R_n} f(z) - f(w_0) - (z - w_0) f'(w_0) \, dz \right|$$

$$\le 4^n \int_{R_n} \left| f(z) - f(w_0) - (z - w_0) f'(w_0) \right| dz$$

$$\le 4^n \; \underbrace{\frac{\varepsilon \Delta}{2^n}}_{\text{bound on integral } *} \cdot \overbrace{\frac{P}{2^n}}^{\text{length of } R_n \text{ perimeter.}}$$

Thus, $\left| \int_R f \right| \le \varepsilon \Delta P, \quad \forall \varepsilon > 0.$

Thus $\left| \int_R f \right| = 0 \Rightarrow \int_R f = 0.$

$\boxtimes$