Homotopy

Def 1 (2.3.6) Suppose $\gamma_0 : [0,1] \to G \subset \mathbb{C}$ and $\gamma_1 : [0,1] \to G \subset \mathbb{C}$ are two continuous curves from z_0 to z_1. We say γ_0 is homotopic with fixed end-points to γ_1 if $\exists$ a continuous function, $H(s,t)$,

$$H : [0,1] \times [0,1] \to G,$$

s.t.

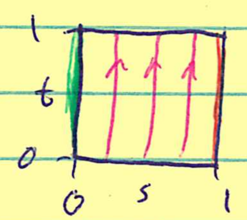
(i) $H(0,t) = \gamma_0(t) \qquad 0 \le t \le 1$

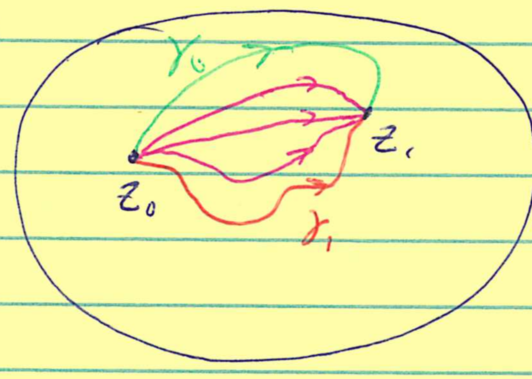
(ii) $H(1,t) = \gamma_1(t) \qquad 0 \le t \le 1$

(iii) $H(s,0) = z_0 \qquad 0 \le s \le 1$

(iv) $H(s,1) = z_1 \qquad 0 \le s \le 1$

Notice, for each $s \in [0,1]$, $\gamma_s(t) = H(s,t)$, is a curve from z_0 to z_1





Def 2 (2.3.7) Suppose $\gamma_0 : [0,1] \to G$ and $\gamma_1 : [0,1] \to G$ are continuous closed curves. We say γ_0 is homotopic as a closed curve to γ_1 if $\exists$ a cont. map, $H(s,t)$

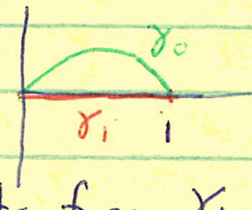
$$H : [0,1] \times [0,1] \to G \text{ with}$$

(i) $H(0,t) = \gamma_0(t) \qquad 0 \le t \le 1$

(ii) $H(1,t) = \gamma_1(t) \qquad 0 \le t \le 1$

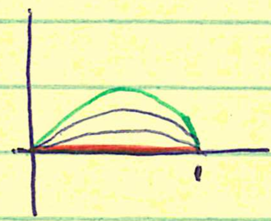
(iii) $H(s,0) = H(s,1) \qquad 0 \le s \le 1.$

Ex 1 Let $\gamma_0(t) = t + t(1-t)i$ and $\gamma_1(t) = t + 0i$.



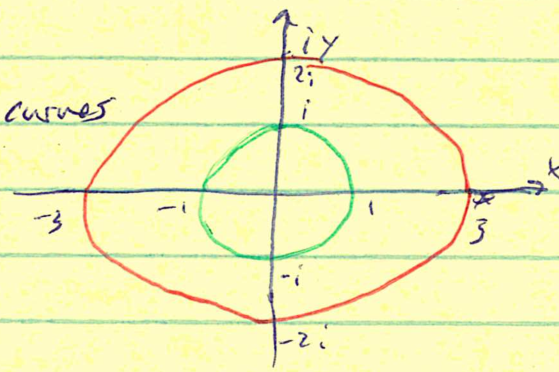
Here is a homotopy fixing end pts from γ_0 to γ_1

$$H(s,t) = (1-s)\left[t + t(1-t)i\right] + s\left[t\right]$$



Ex 2 Let $\gamma_0(t) = e^{2\pi i t}$ and $\gamma_1(t) = 3\cos 2\pi t + 2i \sin 2\pi t$.

Here is a homotopy of closed curves from γ_0 to γ_1.



$H(s,t) = (1-s)\gamma_0(t) + s\gamma_1(t)$.

(In both these $G = \mathbb{C}$.) If in Example 1, G is $\mathbb{C} - \{\frac{1}{2} + \frac{1}{8}i\}$ there is no homotopy taking γ_0 to γ_1 in G. If in Ex 2 G was $\mathbb{C} - \{2\}$, there is no homotopy taking γ_0 to γ_1 in G.

We normally won't bother writing out equations for homotopies.