Technical Lemmas from 2.3 needed in 2.4.

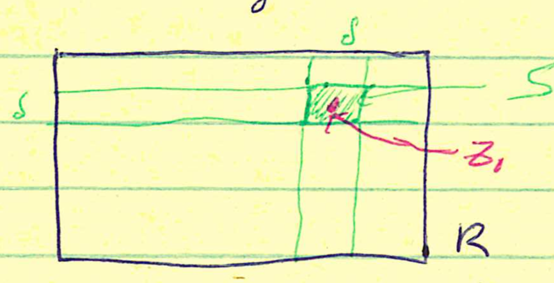
Cauchy's Thm on a Rectangle and a Disk assume f is diff. We can weaken this assumption a bit, by allowing f to be nondiff. at one pt, but that it does not behave too badly.

Lemma 2.3.3. Suppose R is a rectangle with sides parallel to the axes, $f: G \to \mathbb{C}$ is defined on an open set G containing R and its interior, f is diff. on G except at one point z_0 inside R, but with

$$\lim_{z \to z_0} (z - z_0) f(z) = 0.$$

Then we still have $\int_R f = 0$.

Pf Let $\varepsilon > 0$. $\exists \delta > 0$ s.t. $|z - z_0| < \delta \Rightarrow |z - z_0||f(z)| < \varepsilon$. Choose $\delta > 0$ small enough that this holds and so that the square S of side length δ centered at z_0 is inside R.



Now, $\int_R f = \int_S R.$ ~~But~~ $\cancel{\left|\int_S f\right| \le 4\delta \frac{\varepsilon}{|z-z_0|}}$

But $|f(z)| < \frac{\varepsilon}{|z-z_0|}$. Along the perimeter of S, $|z-z_0| \geq \frac{\delta}{2}$

Thus, $|f(z)| < \frac{\varepsilon}{\delta/2} = 2\frac{\varepsilon}{\delta}$, along the perimeter of S.

Thus, $\left|\int_S f\right| \leq \text{perimeter}(S) \cdot \frac{2\varepsilon}{\delta} = \frac{4\delta \cdot 2\varepsilon}{\delta} = 8\varepsilon$.

Since, this holds $\forall \varepsilon > 0$, $\left|\int_S f\right| = 0$.

Thus, $\int_R f = 0$. $\square$

Lemma 2.3.4 R is a rect. with sides parallel to axes. $f: G \to \mathbb{C}$ is cont. on G, an open set contain R and its interior, and that f is diff. on G except at z_0.

If z_0 is outside R we already have this.
If z_0 is inside R, cont at $z_0 \Rightarrow \lim_{z \to z_0} (z-z_0) f(z) = 0$, so we are done.
The other case to check is z_0 is on R. Then the proof is similar to 2.3.3. See textbook.

Thm 2.3.5 The conclusion, ($\int_\gamma f = 0$, F' exists) ~~hold~~ of CTD holds if f is cont. on D and diff. on D - one pt.

Proof is same as CTD (2.3.2) but use the lemmas.