2.4

Def 2.4.1 Let γ be a closed curve in $\mathbb{C}$, $z_0 \in \mathbb{C} \setminus \gamma$. The winding number or index of γ wrt z_0 is

$$I(\gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0}.$$

Examples will be given in class.

Thm 2.4.3 $I(\gamma, z_0) \in \mathbb{Z}$.

Pf Let $g(t) = \displaystyle\int_a^t \frac{\gamma'(s)}{\gamma(s) - z_0}\, ds$. ($\gamma : [a,b] \to \mathbb{C}$, $\gamma(a) = \gamma(b)$.)

Let $h(t) = e^{-g(t)} \cdot (\gamma(t) - z_0)$. We will show $h(t)$ is a constant.

$$h'(t) = -g'(t) e^{-g(t)} (\gamma(t) - z_0) + e^{-g(t)} \gamma'(t)$$

$$= \frac{-\gamma'(t)}{\cancel{\gamma(t) - z_0}} e^{-g(t)} \cancel{(\gamma(t) - z_0)} + e^{-g(t)} \gamma'(t)$$

$$= -\gamma'(t) e^{-g(t)} + e^{-g(t)} \gamma'(t) = 0.$$

Thus $h(a) = h(b)$. Thus $e^{-g(a)} (\gamma(a) - z_0) = e^{-g(b)} (\gamma(b) - z_0)$.

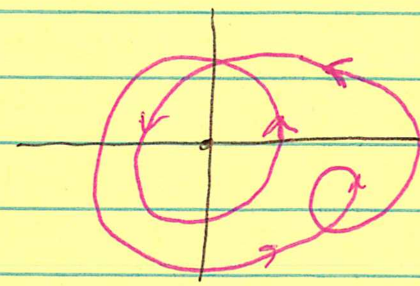
Since $\gamma(a) = \gamma(b)$, we have $e^{-g(a)} = e^{-g(b)}$. But $g(a) = 0$.
Thus, $e^{-g(b)} = 1 \Rightarrow -g(b) = 2\pi n i$.

Now $I(\gamma, z_0) = \dfrac{g(b)}{2\pi i} = -n \in \mathbb{Z}$. $\square$

Thm (2.4.4) Let f be diff. in a region A. Let γ be a closed curve in A that is homotopic to a point. Let $z_0 \in A \setminus \gamma$. Then

$$\int_\gamma \frac{f(z)}{z-z_0}\,dz = 2\pi i\, I(\gamma, z_0)\, f(z_0).$$

Ex Compute $\int_\gamma \frac{e^z}{z}\,dz$ where γ is



Sol $\int_\gamma \frac{e^z}{z}\,dz = 2\pi i\, I(\gamma, 0)\, e^0$

$$= 2\pi i \cdot 2 \cdot 1 = 4\pi i.$$

Pf Let $g(z) = \begin{cases} \frac{f(z)-f(z_0)}{z-z_0} & \text{if } z \neq z_0 \\ f'(z_0) & \text{if } z = z_0. \end{cases}$

Then g is cont. on A and diff. on $A - \{z_0\}$.

Thus, $0 = \int_\gamma g(z)\,dz = \int_\gamma \frac{f(z)-f(z_0)}{z-z_0}\,dz$

$$= \int_\gamma \frac{f(z)}{z-z_0}\,dz - f(z_0)\int_\gamma \frac{1}{z-z_0}\,dz$$

$$= \int_\gamma \frac{f(z)}{z-z_0}\,dz - f(z_0)\, 2\pi i\, I(\gamma, z_0).$$

Thus, $\int_\gamma \frac{f(z)}{z-z_0}\,dz = 2\pi i\, I(\gamma, z_0)\, f(z_0)$. $\boxtimes$

See Ex 2.4.14 in textbook (p 156-7)

Thm 2.4.5 Suppos γ is a curve in $\mathbb{C}$ and $g(z)$ is continuous also γ. Let

$$G(z) = \frac{1}{2\pi i} \int_\gamma \frac{g(w)}{w-z}\, dw.$$

Then $G(z)$ is ~~analytic~~ infinitely diff for $z \notin \gamma$ with

$$G^{(k)}(z) = \frac{k!}{2\pi i} \int_\gamma \frac{g(w)}{(w-z)^{k+1}}\, dw.$$

Rough Proof

$$\frac{d}{dz} G(z) = \frac{1}{2\pi i} \frac{d}{dz} \int_\gamma \frac{g(w)}{w-z}\, dw = \frac{1}{2\pi i} \int_\gamma \frac{\partial}{\partial z}\left(\frac{g(w)}{w-z}\right) dw$$

$$= \frac{1}{2\pi i} \int_\gamma \frac{g(w)}{(w-z)^2}\, dw.$$

$$\frac{d^2}{dz^2} G(z) = \frac{1}{2\pi i} \int_\gamma \frac{\partial}{\partial z}\left(\frac{g(w)}{(w-z)^2}\right) dw = \frac{2}{2\pi i} \int_\gamma \frac{g(w)}{(w-z)^3}\, dw$$

$$\frac{d^3}{dz^3} G(z) = \frac{2}{2\pi i} \int_\gamma \frac{\partial}{\partial z}\left(\frac{g(w)}{(w-z)^3}\right) dw = \frac{3 \cdot 2}{2\pi i} \int_\gamma \frac{g(w)}{(w-z)^3}\, dw$$

etc.

The "real" proof at the end of this section.

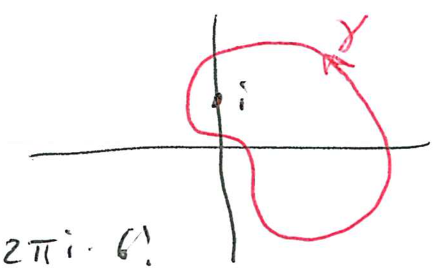
Thm 2.4.6 (Cauchy's Integral Formula for Derivatives)

Let f be analytic on $A \subset \mathbb{C}$ (A open). Let $z_0 \in A$ and let γ be a closed curve homotopic to a pt in A, $z_0 \notin \gamma$. Then all derivatives of f exist on A, and

$$f^{(k)}(z_0) \cdot I(\gamma, z_0) = \frac{k!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z_0)^{k+1}}\, dw, \quad k = 1,2,3,\ldots$$

Pf Use Thm 2.4.5. See textbook.

Example Compute $\int_\gamma \frac{\cos(w)}{(w-i)^7}\, dw$, where γ is as shown.



Solution Let $k=6$. $6! = 720$

$$\int_\gamma \frac{\cos(w)}{(w-i)^7}\, dw = \frac{d^6}{dz^6}\cos(z)\Big|_{z=i} \cdot \underbrace{I(\gamma, i)}_{=1} \cdot 2\pi i \cdot \underbrace{6!}_{=720}$$

$$\frac{d^6}{dz^6}\cos(z) = -\cos(z). \qquad -\cos(i) = -\frac{e^{ii} + e^{-ii}}{2} = -\frac{1}{2}\left(\frac{1}{e} + e\right).$$

Thus,
$$\int_\gamma \frac{\cos(w)}{(w-i)^7}\, dw = -\frac{1}{\cancel{2}}\left(\frac{1}{e}+e\right) \cdot 1 \cdot \cancel{2}\pi i \cdot 720 = -\left(\frac{1}{e}+e\right) 720\pi i.$$

Thm 2.4.7 (Cauchy's Inequalities) Let A be an open conn'd set in $\mathbb{C}$. Let f be analytic on A. Let γ be a circle of radius R center z_0 s.t. γ and the disk it bounds are inside A. Suppose $|f(z)| \le M$ $\forall z \in \gamma$. Then

$$\left| f^{(k)}(z_0) \right| \le \frac{k!}{R^k} M.$$

Pf Since $I(\gamma, z_0) = 1$, we have

$$f^{(k)}(z_0) = \frac{k!}{2\pi i} \int_\gamma \frac{f(w)}{(w - z_0)^{k+1}} \, dw.$$

Thus,
$$\left| f^{(k)}(z_0) \right| = \frac{k!}{2\pi} \left| \int_\gamma \frac{f(w)}{(w - z_0)^{k+1}} \, dw \right|$$

$$\le \frac{k!}{2\pi} \int_\gamma \left| \frac{f(w)}{(w - z_0)^{k+1}} \right| dw$$

$|w - z_0| = R$ along γ and $|f(w)| \le M$ along γ.

Thus
$$\left| f^{(k)}(z_0) \right| \le \frac{k!}{2\pi} \frac{M}{R^{k+1}} \cdot \text{length}(\gamma) = 2\pi R$$

$$= \frac{k! \, M}{R^k}, \text{ as claimed.} \quad \square$$

Thm 2.4.8 (Liouville's Thm) If f is entire and there is a constant $M \geq 0$ s.t. $|f(z)| \leq M$ for all $z \in \mathbb{C}$, then f is a constant.

Pf Pick any $z \in \mathbb{C}$. We claim $f'(z) = 0$. Thus f is a constant. By Cauchy's Inequalities (2.4.7) with $k=1$ we have, for any $R > 0$,

$$|f'(z)| \leq \frac{M}{R}.$$

$\forall \varepsilon > 0$, $\exists R > 0$ s.t. $\frac{M}{R} < \varepsilon$. Thus $|f'(z)| < \varepsilon$, $\forall \varepsilon > 0$. Hence $|f'(z)| = 0$. $\square$

Ex $|\sin z|$ must be unbdd over $z \in \mathbb{C}$, since we know $\sin z$ is not constant and is entire.

Thm (2.4.4) (Fundamental Thm of Algebra) Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots a_1 z + a_0$ be a poly. with complex coeff's. Assume $n \geq 1$ and $a_n \neq 0$. Then $\exists\, z_0 \in \mathbb{C}$ s.t. $p(z_0) = 0$.

Pf The proof is by contradiction. Suppose $p(z)$ has no solutions in $\mathbb{C}$. We will use Liouville's Thm to show it is a constant function, which it clearly is not.

Let $f(z) = \frac{1}{p(z)}$. Then $f(z)$ is entire. We claim it is bdd and hence constant.

$$f(z) = \frac{1}{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0} = \frac{\frac{1}{z^n}}{a_n + a_{n-1}\left(\frac{1}{z}\right) + a_{n-2}\left(\frac{1}{z^2}\right) + \cdots + a_0\left(\frac{1}{z^n}\right)}$$

Along any path in $\mathbb{C}$ with $|z| \to \infty$ the limit of $f(z)$ is therefore zero. Thus, $\forall\, \varepsilon > 0$, $\exists\, R > 0$ s.t. for $|z| > R$ we have

$$|f(z)| < \varepsilon.$$ Let $\varepsilon = 1$ and R as stated.

On the closed disk $\overline{D(0,R)}$, $|f(z)|$ is bounded because $|f(z)|$ is cont. and the closed disk is compact. Let M be a bound. Then the max of M and 1 is a bound for $|f(z)|$ over all of $\mathbb{C}$.

Thus $f(z)$ is a constant which implies $p(z)$ is a constant.

Since this is a contradiction, $\exists\, z_0 \in \mathbb{C}$ s.t. $p(z_0) = 0$. $\square$