Section 2.4 Outline

Def (2.4.1) Winding number, or index, of a closed curve γ with respect to a point z_0, not on γ,

$$I(\gamma, z_0) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-z_0}\,dz.$$

Thm (2.4.3) $I(\gamma, z_0) \in \mathbb{Z}$.

Thm (2.4.4) Under suitable conditions $\int_\gamma \frac{f(z)}{z-z_0}\,dz = 2\pi i\, I(\gamma, z_0)\cdot f(z_0)$

Thm (2.4.5) Under suitable conditions, if we define

$$G(z) = \frac{1}{2\pi i}\int_\gamma \frac{g(w)}{w-z}\,dw$$

Then all derivatives of $G(z)$ exist and

$$G^{(k)}(z_0) = \frac{k!}{2\pi i}\int_\gamma \frac{g(w)}{(w-z)^{k+1}}\,dw, \quad k=1,2,3\ldots$$

Thm (2.4.6) If f' exists on $A \subset \mathbb{C}$, then all $f^{(k)}$ exist and

$$I(\gamma, z_0)\cdot f^{(k)}(z_0) = \frac{k!}{2\pi i}\int_\gamma \frac{f(w)}{w-z_0}\,dw, \quad k=1,2,3\ldots$$

~~γ is any closed curve homotopic to z_0~~

Thm 2.4.8 (Liouville's Thm). If f is entire and there is a constant M s.t. $|f(z)| \le M$ for all $z \in \mathbb{C}$, then f is a constant!

Thm 2.4.9 (Fundamental Thm of Alg). Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ where $a_i \in \mathbb{C}$. Then $\exists\, z_0 \in \mathbb{C}$ s.t. $p(z_0) = 0$.

Thm 2.4.10 (Morera's Thm) Let f be cont. on A, open conn'd, and suppose $\int_\gamma f = 0$ for every closed curve in A.

Then f is analytic on A and $f = F'$ for some analytic F on A.

\+ more fun with log.