2.5 Part II: Harmonic Functions

Recall If $u: \mathbb{R}^2 \to \mathbb{R}$ and $\nabla^2 u = 0$, u is harmonic.

If $f: \mathbb{C} \to \mathbb{C}$ is analytic and $u = \operatorname{Re} f$, then u is harmonic, also $v = \operatorname{Im} f$.

Prop. 2.5.8 Show given a harmonic u, $\exists f: \mathbb{C} \to \mathbb{C}$ analytic with $u = \operatorname{Re} f$. Letting $v = \operatorname{Im} f$, v is a harmonic conjugate of u. We did examples back in Section 1.5.

It turns out harmonic function share some of the interesting properties of analytic functions:

Thm 2.5.9 (Mean Value Property for Harmonic Functions)
Let u be harmonic on an open set A containing a circle of radius r, center $z_0 = x_0 + iy_0$ and its interior. Then

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta.$$

Pf $\exists \varepsilon > 0$, s.t. $D(z_0, r+\varepsilon) \subset A$. (See Example 1.4.27)
Let f be analytic on A with $\operatorname{Re} f = u$. We know

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta.$$

Take real part of both sides. $\square$

Because of this the MMP works for harmonic functions
See Thm's 2.5.10 (Local) & 2.5.11 (Global).

An application of this is the Dirichlet Problem.

Let $A \subset \mathbb{R}^2$ be an open, bdd, conn'd set.
Let u_0 be a cont. function defined on bd(A).
Find a harmonic function u on A that is
cont. on cl(A) and $= u_0$ on bd(A).

Thm 2.5.12 show solutions are unique when they exist.

Thm 2.5.13 gives a solution when A = disk = $D(0,r)$.
Note: Thm 5.1.4 (pg 321) Riemann Mapping Thm shows
many regions can be mapped to a disk in a "nice" way.

Thm 2.5.13 Assume u is cont. on $cl(D(0,r))$ and is
harmonic on $D(0,r)$. Let $\rho < r$, $\phi \in [0, 2\pi)$. Then
knowing u for $|z| = r$, determines u inside $D(0,r)$
via

$$u(\rho e^{i\phi}) = \frac{r^2-\rho^2}{2\pi} \int_0^{2\pi} \frac{u(re^{i\theta})}{r^2 - 2r\rho\cos(\phi-\theta)+\rho^2}\, d\theta$$

In other words, if you know U along the boundary
circle, you can compute it inside.

$\underline{E_I}$ If $u(re^{i\theta})=0$, then $u(\rho,\phi)=0$, $\rho \le r$.

$\underline{E_I}$ Exercise 12 (a) show if $u(re^{i\theta})=1$, $u(\rho,\phi)=1$.

(b) ~~cos θ~~ $u(1\cdot e^{i\theta})=\cos x$ is given.

Then $u(\rho e^{i\phi})=\rho\cos\phi$.