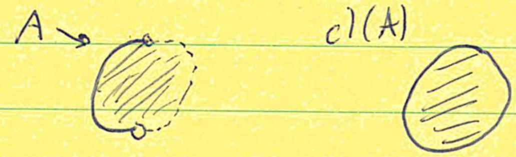
2.5

First we report some notation and facts about closed subsets of $\mathbb{C}$.

Def The closure of a set $A \subset \mathbb{C}$, denoted $\bar{A}$ or $cl(A)$, is A unioned with the set of points that are limits of convergent sequences in A.

Ex



Def The boundary of $A \subset \mathbb{C}$ is

$$bd(A) = cl(A) \cap cl(\mathbb{C} - A).$$

Facts (Prop 2.5.4 and Exam. 2.5.16)

(i) $A \subset cl(A)$

(ii) A is closed $\Leftrightarrow A = cl(A)$.

(iii) $A \subset C$, and C is closed, $\Rightarrow cl(A) \subset C$

(iv) $cl(A)$ is closed

(v) $cl(A) = A \cup bd(A)$

Thm (2.5.2) (Mean Value Property) Let f be analytic on and inside a circle γ of radius r, center z_0. Then

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta.$$

pf Let $\gamma = z_0 + re^{i\theta}$, $0 \le \theta \le 2\pi$.
By 2.4.4

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\,dz$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{\cancel{z_0} + re^{i\theta} - \cancel{z_0}}\, rie^{i\theta}\,d\theta$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} f(z_0 + re^{i\theta})\, i\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta.$$

Fact (From Example 1.5.21) Let f be analytic on an open conn'd set $A \subset \mathbb{C}$. Suppose $|f|$ is a constant. Then f is a constant.

Pf Let $f = u + iv$. Then $|f|^2 = u^2 + v^2 = c$, a constant. If $c = 0$, we are done since $|f| = 0 \Rightarrow f = 0$. Assume $c \neq 0$.

Now $\quad 2uu_x + 2vv_x = 0$ and $2uu_y + 2vv_y = 0$.
But we know $u_x = v_y$ and $u_y = -v_x$. Thus,

$$uu_x - vu_y = 0 \quad \text{and} \quad vu_x + uu_y = 0.$$

In matrix form we have $\begin{bmatrix} u & -v \\ v & u \end{bmatrix}\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. $\star$

But $\det \begin{bmatrix} u & -v \\ v & u \end{bmatrix} = u^2 + v^2 = c \neq 0$. Thus $\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is the unique solution to $\star$.

Thus $f' = u_x - iu_y = 0$. Thus, f is a constant. $\square$

Maximum Modulus Principle

Thm 2.5.6 Let A be an open, connected, bounded set in $\mathbb{C}$. Suppose $f: cl(A) \to \mathbb{C}$ is analytic on A and continuous on $cl(A)$. By continity and compactness, we know $|f|$ has a (finite) max on $cl(A)$.

(i) This max will be obtained on $bd(A)$

(ii) If it is also obtained inside A, then f must be a constant of $cl(A)$.

First we establish a local version.

Thm 2.5.1. Let f be analytic on an open set A and suppose $|f|$ has a relative maximum at $z_0 \in A$. Then $\exists r > 0$ s.t. f is a constant on $D(z_0, r) \subset A$.

Outline of Proof By 2.5.2 $f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta$ for ~~some~~ $r > 0$. as long as $D(z_0, r) \subset A$.

Suppose $\exists z_1$ in $D(z_0, r)$ s.t. $|f(z_1)| < |f(z_0)|$. Let $r'e^{i\theta'} = z_1 - z_0$. Then 2.5.2 will fail

$|f(z_0 + r'e^{i\theta})|$ can never exceed $|f(z_0)|$.

If $|f(z_0 + r'e^{i\theta'}| < |f(z_0)|$, then $\left|\frac{1}{2\pi}\int_0^{2\pi} f(z_0 + r'e^{i\theta})\, d\theta\right|$ will be too small. I'll draw pictures in class.

Textbook has details. Once $|f|$ is constant, f is.

Proof of 2.5.6. (MMP)

Suppose $\exists\, a \in A$ where $|f(a)| = $ max of $|f(z)|$ over $z \in cl(A)$.

Let $A_1 = \{z \in A \mid f(z) = f(a)\}$. We claim $A_1 = A$.

A_1 is open: Let $z \in A_1$. There is a disk $D(z,r) \subset A$ s.t. f is constant. Since $f(z) = f(a)$, then $f(w) = f(a)$ $\forall\, w \in D(w,r)$.
Thus $D(z,r) \subset A_1$

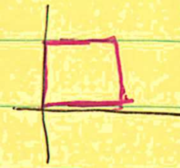
Let $A_2 = A - cl(A_1)$. A_2 is open $(A_2 = A \cap (\mathbb{C} - cl(A_1)))$.

Clearly $A_1 \cap A_2 = \phi$. Claim $A \subset A_1 \cup A_2$. Pf: Suppose $z \in A$ but not A_2. Then $z \in cl(A_1)$. $\Rightarrow f(z) = f(a)$.
Thus $z \in A_1$. Hence $A \subset A_1 \cup A_2$.

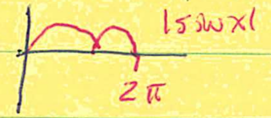
But A is connected. Thus, $A_2 = \phi$ or we have a separation of A. Thus $A \subset A_1$. Thus $A = A_1$. $\square$

Ex Find max of $|\sin z|$ over the square $[0,2\pi]\times[0,2\pi] \subset \mathbb{C}$.

Sol. By the MMP we only need to check the bdy.

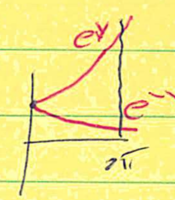


Let $y=0$, $0 \le x \le 2\pi$, $z = x+iy$.
Thus, $|\sin(z)| = |\sin(x)|$ which has a max of 1 at $x = \frac{\pi}{2}, \frac{3\pi}{2}$.



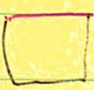
Let $x=0$, $0 \le y \le 2\pi$.

$$|\sin(z)| = |\sin(iy)| = \left|\frac{e^{iiy} - e^{-iiy}}{2i}\right| = \left|\frac{e^{-y} - e^{y}}{2i}\right| = \frac{e^{y} - e^{-y}}{2}$$



Max is at $y = 2\pi$, and is $\dfrac{e^{2\pi} - e^{-2\pi}}{2}$.

Let $x = 2\pi$, $0 \le y \le 2\pi$. Since $\sin(2\pi + iy) = \sin(iy)$, the result is the same.



Now, let $y = 2\pi$ and $0 \le x \le 2\pi$.

$$|\sin z| = |\sin(x + 2\pi i)| = \left|\frac{e^{ix-2\pi} - e^{-ix+2\pi}}{2i}\right|$$

$$= \frac{1}{2}\left| e^{-2\pi}(\cos x + i\sin x) - e^{2\pi}(\cos x - i \sin x)\right|$$

$$= \frac{1}{2}\left| (e^{-2\pi} - e^{2\pi})\cos x + i(e^{-2\pi} + e^{2\pi})\sin x\right|$$

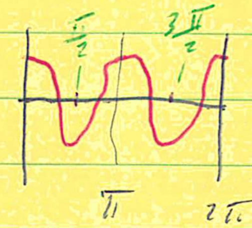
$$= \frac{1}{2}\left((e^{-2\pi} - e^{2\pi})^2 \cos^2 x + (e^{-2\pi} + e^{2\pi})^2 \sin^2 x \right)^{1/2}$$

$$= \frac{1}{2}\left((e^{-4\pi} - 2 + e^{4\pi}) \cos^2 x + (e^{-4\pi} + 2 + e^{4\pi}) \sin^2 x \right)^{1/2}$$

$$= \frac{1}{2}\left(e^{-4\pi} + e^{4\pi} + 2(\sin^2 x - \cos^2 x) \right)^{1/2}$$

$$= \frac{1}{2}\left(e^{-4\pi} - 2\cos(2x) + e^{4\pi} \right)^{1/2}.$$

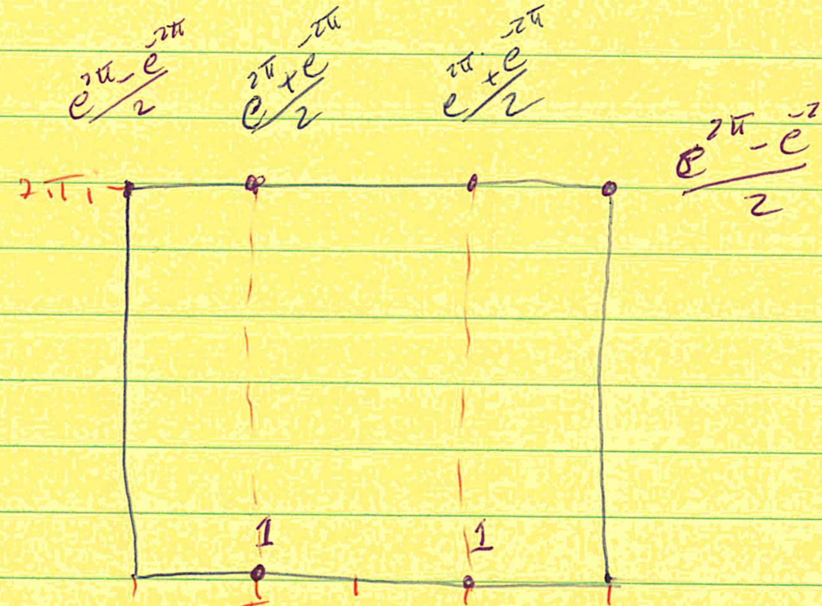
Max occurs when $\cos(2x) = -1$, $\frac{\pi}{2}, \frac{3\pi}{2}$



At these points,

$$|\sin z| = \frac{1}{2}(e^{-2\pi} + e^{2\pi}).$$

Summary



Thus, max $|\sin z|$ is $\frac{e^{2\pi} + e^{-2\pi}}{2}$ at $\frac{\pi}{2} + 2\pi i$ and $\frac{3\pi}{2} + 2\pi i$

Ex Find max of $|e^{z^2}|$ over the closed unit disk, $|z| \le 1$.

Sol. By the MMP we need only check the bdy, $z = e^{i\theta}$.

$$\left| e^{e^{2i\theta}} \right| = \left| e^{\cos 2\theta + i \sin 2\theta} \right| = e^{\cos(2\theta)}.$$

Max occurs when $\cos(2\theta) = 1$; $\theta = 0, \pi, 2\pi$.

The max is $e^1 = e$.