3.1 Ch 3: Series and Sequences

Def $\lim_{n\to\infty} z_n = z_0$, or $z_n \to z_0$, means

$\forall \varepsilon > 0, \exists N$ s.t. $n > N \Rightarrow |z_0 - z_n| < \varepsilon$.

$\sum_{k=1}^{\infty} a_k = S$ means $\lim_{n\to\infty} \sum_{k=1}^{n} a_k = S$.

Def A seq $(z_n)_{n=1}^{\infty}$ is Cauchy if $\forall \varepsilon > 0, \exists N$ s.t.

$m, n \geq N \Rightarrow |z_m - z_n| < \varepsilon$. (That is, the terms, z_n, bunch up.)

Thm A seq in $\mathbb{C}$ (or any $\mathbb{R}^n$) converges if and only if it is Cauchy.

This can be reframed for series as follows.

Def A series $\sum_{k=1}^{\infty} a_k$ is Cauchy if $\forall \varepsilon > 0, \exists N$ s.t.

$n \geq m > N \Rightarrow \left| \sum_{k=m}^{n} a_k \right| < \varepsilon$.

Thm A series in $\mathbb{C}$ (or any $\mathbb{R}^n$) converges if and only if it is Cauchy.

Fact If $\sum a_k$ converges, then $a_k \to 0$.

Pf Use $n = m = k$.

Def A seq $\sum a_k$ converges absolutely if $\sum |a_k|$ converges.

Prop 3.1.2 If $\sum a_k$ conv. abs., it converges.

Pf Let $\varepsilon > 0$. Let N be s.t. $n > N, p > 0 \Rightarrow$

$$\sum_{k=n}^{n+p} |a_k| < \varepsilon.$$

Then $$\left| \sum_{k=n}^{n+p} a_k \right| \le \sum_{k=n}^{n+p} |a_k| < \varepsilon. \qquad \boxtimes$$

Basic Facts (Prop 3.1.3)

(i) Geom series: $|r| < 1 \Rightarrow \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$.

$|r| \ge 1 \Rightarrow \sum_{n=0}^{\infty} r^n$ diverges (does not converge)

(ii) Comparison Tests: If $\sum_{k=1}^{\infty} b_k$ converges and $0 \le a_k \le b_k$,

then $\sum_{k=1}^{\infty} a_k$ converges. ($a_k, b_k \in \mathbb{R}$)

If $\sum_{k=1}^{\infty} b_k$ diverges and $0 \le b_k \le a_k$, then

$\sum_{k=1}^{\infty} a_k$ diverges.

(iii) p-Series: $\sum_{n=1}^{\infty} n^{-p}$ converges for $p > 1$, div for $p \le 1$. ($p \in \mathbb{R}$)

(iv) Ratio Test: If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ conv. abs.

If $L > 1$, $\sum_{n=1}^{\infty} a_n$ diverges.

(v) Root Test: If $\lim_{n\to\infty} (|a_n|)^{1/n} = L < 1$, then $\sum_{n=1}^{\infty} a_n$ conv. abs.

If $L > 1$, $\sum_{n=1}^{\infty} a_n$ div.

Pf: See textbook. Covered in Math 352.

Sequences and Series of Functions

Def (31.9) Let $A \subset \mathbb{C}$ and $f_n : A \to \mathbb{C}$, $n = 1, 2, 3, \ldots$
If for each point $z \in A$, $f_n(z)$ converges we can define $f(z) = \lim_{n\to\infty} f_n(z)$. We say f_n converges to f pointwise.

A seq of functions, $f_n : A \to \mathbb{C}$, converges uniformly to f if

$$\forall \varepsilon > 0, \exists N \text{ s.t. } n > N \Rightarrow |f_n(z) - f(z)| < \varepsilon, \forall z \in A.$$

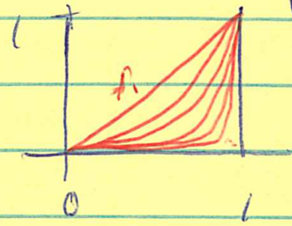
(N depends on ε, but not on z)

Uniform conv. $\Rightarrow$ pt-wise conv., but not the reverse.

Similar definition are made for series of functions.

Ex (in $\mathbb{R}$) Let $f_n(x) = x^n$ on $[0,1]$.

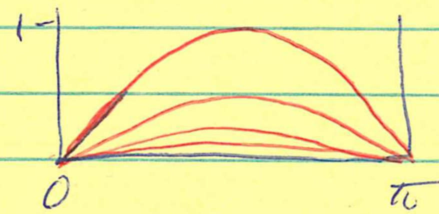
Then $f_n(x) \to f(x) = \begin{cases} 0 & x \in [0,1), \\ 1 & x = 1. \end{cases}$



limit is pointwise, but not uniform.

Ex (in $\mathbb{R}$) Let $f_n(x) = \dfrac{\sin x}{n}$ on $[0, \pi]$

Then $f_n(x) \to 0$ ptwise and uniformly.



Thm 3.1.5 (Cauchy Criterion)

(i) A seq $f_n(z)$ converges uniformly on A if and only if

$\forall \varepsilon > 0, \exists N$ s.t. $n \geq N$ $p > 0 \Rightarrow |f_n(z) - f_{n+p}(z)| < \varepsilon$ $\forall z \in A$.

(ii) A series $\sum_{k=1}^{\infty} g_k(z)$ conv. uniformly on A iff

$\forall \varepsilon > 0$ $\exists N$ s.t. $n \geq N$ $p > 0$, $\Rightarrow \left| \sum_{k=n}^{n+p} g_k(z) \right| < \varepsilon$, $\forall z \in A$.

Pf A simple reworking of the definitions.

Prop 3.1.6 (i) If (f_n) consists of cont. functions on $A \subset \mathbb{C}$, and $f_n \to f$ unif, then f is continuous.
(ii) Likewise for series.

Pf of (i) Let $z_0 \in A$, let $\varepsilon > 0$.

$\exists N$ s.t. $n \geq N \Rightarrow |f_n(z) - f(z)| < \frac{\varepsilon}{3}$, $\forall z \in A$.

Since f_N is continuous, $\exists \delta > 0$ s.t. $|f_N(z) - f_N(z_0)| < \varepsilon/3$ whenever $|z - z_0| < \delta$.

Now, $|f(z) - f(z_0)| =$

$$|f(z) - f_N(z) + f_N(z) - f_N(z_0) + f_N(z_0) - f(z_0)|$$

$$\leq |f(z) - f_N(z)| + |f_N(z) - f_N(z)| + |f_N(z_0) - f(z_0)|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Thus, f is cont at z_0. This works for any $z_0 \in A$. $\square$

Thm (3.1.7) (Weierstrass M Test)

Let g_n be a seq. of functions, $g_n : A \to \mathbb{C}$.
Suppose $\exists$ a seq of real numbers, $M_n \geq 0$ s.t.

(i) $|g_n(z)| \leq M_n$, $\forall z \in A$, and

(ii) $\sum_{n=1}^{\infty} M_n$ converges.

Then $\sum_{n=1}^{\infty} g_n$ converges absolutely and uniformly on A.

<u>Pf</u> Let $\varepsilon > 0$. $\exists N$ s.t. $n \geq N \Rightarrow \sum_{k=n+1}^{n+p} M_k < \varepsilon$, $\forall p \geq 0$.

Now, $n \geq N \Rightarrow$

$$\left| \sum_{k=n+1}^{n+p} g_k(z) \right| \leq \sum_{k=n+1}^{n+p} |g_k(z)| \leq \sum_{k=n+1}^{n+p} M_k < \varepsilon, \ \forall z \in A.$$

Thus $\sum |g_k|$ converges and we see the conv. is uniform.

Hence $\sum g_k$ converges absolutely and uniformly.

Prop 3.1.9 Let γ be a curve in a region $A \subset \mathbb{C}$, (open conn'd).
Let f_n be continuous and defined on $\gamma([a,b])$.

(i) Let $f_n \to f$ uniformly on $\gamma([a,b])$.
Then $$\int_\gamma f_n \to \int f.$$

(ii) Let $\sum f_n \to f$ uniformly on $\gamma([a,b])$.

Then $$\int_\gamma \sum f_n = \sum \int_\gamma f_n.$$

Pf (i) We know f is cont. by Prop 3.1.6. So $\int_\gamma f$ is defined.
Let $\varepsilon > 0$. Choose N s.t. $n \geq N \Rightarrow |f_n(z) - f(z)| < \frac{\varepsilon}{l(\gamma)}$
$\forall z \in \gamma$. Then by Prop 2.1.6

$$\left| \int_\gamma f_n - \int_\gamma f \right| = \left| \int_\gamma f_n - f \right| \leq \int_\gamma |f_n(z) - f(z)| |dz|$$

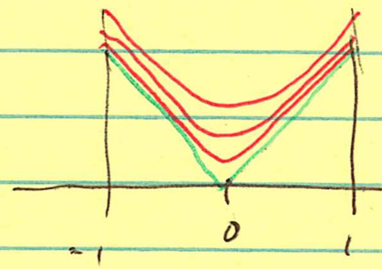
$$\leq \frac{\varepsilon}{l(\gamma)} \cdot l(\gamma) = \varepsilon.$$

(ii) Proof is the same. Just apply (i) to the seq. of partial sums, $\sum_{k=1}^{n} f_k$.

Ex Let $f_n : [-1,1] \to \mathbb{R}$ be given by $f_n(x) = \sqrt{\frac{1}{n} + x^2}$.

Each f_n is cont. and diff. $f_n(x) \to |x|$ uniformly.

But while $|x|$ is cont., it is not diff. at $x=0$.



However, for analytic functions this cannot happen!

Thm 3.1.8 (Analytic Convergence Thm)

(i) Let A be an open set in $\mathbb{C}$. Let f_n be a seq. of analytic functions from A to $\mathbb{C}$. If $f_n \to f$ uniformly on every closed disk in A, then f is analytic on A, and $f_n' \to f'$ pointwise on A and uniformly on every closed disk in A.

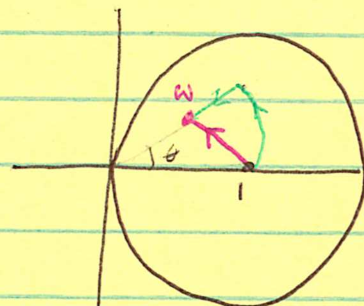
(ii) Likewise for series! Let A and f_n be as above. If $f = \sum f_n$ converges uniformly on every closed disk in A, then f is analytic on A and $f' = \sum f_n'$ pointwise on A and uniformly on every closed disk in A.

Pf See textbook.

Ex 3.1.14 Show $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}$ conv. unif and abs. to $\log(z+1)$ on $D(0,1)$. Here we use $\log re^{i\theta} = \ln r + i\theta$, $-\pi < \theta < \pi$.

Sol. $z \in D(0,1) \Rightarrow z+1 \in D(1,1)$. Let $w \in D(1,1)$.

Let γ be a straight line segment from 1 to w. Then $\gamma \subset D(1,1)$.



Then $\int_\gamma \frac{1}{z}\, dz = \log(w)$

But, we can change the path ~~to~~ ~~the~~ an arc $(r=1)$ followed by a ray (θ fixed) as shown in green. Now

$$\log w = \int_\gamma \frac{1}{z}\, dz = \int_0^\theta \frac{1}{e^{i\phi}}\, ie^{i\phi}\, d\phi + \int_1^{|w|} \frac{1}{re^{i\theta}}\, \underbrace{e^{i\theta}}_{\frac{d(re^{i\theta})}{dr}}\, dr$$

$$= i\theta + \ln|w|.$$

Now change variable, let $q = z-1$. $dq = dz$.

So,
$$\log w = \int_\mu \frac{1}{q+1}\, dq = \int_\mu \frac{1}{1-(-q)}\, dq$$

where μ is line seg from 0 to $w-1$ in $D(0,1)$.

Ex 3.1.12 Show that $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ on $D(0,1)$.

Show that the convergence is absolute and uniform on every closed disk $D_r = cl(D(0,r))$, $0<r<1$.

Solution Let $z \in D(0,1)$. Pick $r \in (|z|, 1)$.
Then $|z^n| < r^n$, $\sum r^n$ converges by Geom series test.
Using $M_n = r^n$, the W. M-test shows $\sum z^n$ conv. abs. and uniformly in D_r.

The formula for the limit is just the same as the Geom Series formula:

$$1 - z^{n+1} = (1-z)(1+z+z^2+\cdots+z^n)$$

Thus,

$$\left| \frac{1}{1-z} - \sum_{k=0}^{n} z^k \right| = \left| \frac{1}{1-z} - \frac{1-z^{n+1}}{1-z} \right|$$

$$= \left| \frac{z^{n+1}}{1-z} \right| \leq \frac{r^{n+1}}{1-r} \longrightarrow 0.$$

Ex 3.1.13 Show that the series $\sum_{n=0}^{\infty} n z^{n-1} = \frac{1}{(1-z)^2}$ on $D(0,1)$.

The conv. is abs. and unif on every closed disk in $D(0,1)$.

Sol. Let $B \subset D(0,1)$ be any closed disk. $\exists\, r \in (0,1)$ s.t. $B \subset D_r \subset D(0,1)$.

Thus, by last example, $\sum_{n=0}^{\infty} z^n$ conv. abs and unif on B to $\frac{1}{1-z}$.

Use term-by-term differentiation to get

$$\left(\sum_{n=0}^{\infty} z^n\right)' = \left(\frac{1}{1-z}\right)'$$

$$\Rightarrow \sum_{n=0}^{\infty} n z^{n-1} = \frac{1}{(1-z)^2}$$, uniformly on B by Thm 3.18 ii.

To get abs. convergence use a comparison with $\sum_{n=0}^{\infty} n r^{n-1}$.

But we have $\dfrac{1}{1-(-q)} = \sum_{n=0}^{\infty} (-q)^n$.

Thus

$$\log w = \int_{\mu} \sum (-q)^n \, dq = \sum \int_{\mu} (-q)^n \, dq$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(w-1)^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(w-1)^n}{n},$$

Which is equivalent to

$$\log(z+1) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}.$$

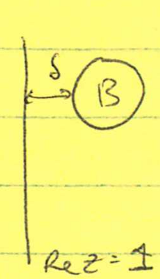
You can us W. M-test to get convergence to be uniform and absolute. $\blacksquare$

__Ex 3.1.15__ $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ (should remind you of p-series $\sum \frac{1}{n^p}$)

Show analytic on $A = \{z \mid \mathrm{Re}\, z > 1\}$.
Find conv. series for $\zeta'(z)$ on A.

__Solution__

By Prop 1.6.4 n^{-z} is entire.
To use Thm 3.1.8 (Analytic Conv. Thm) we need to show the conv. is unif. on every closed disk in A. Let $B \subset A$ be a closed disk.
Let $\delta = \mathrm{dist}(\mathbb{C} \setminus A, B)$.



We want to use W. M-Test.
Need to find bounds on $|n^{-z}|$.

$$|n^{-z}| = |e^{-z \log n}| = |e^{-x \log n - iy \log n}| = e^{-x \log n} = n^{-x}$$

But for $z \in B$, $x \geq 1+\delta$, so $n^{-x} \leq n^{-(1+\delta)}$.
Let $M_n = n^{-(1+\delta)}$.

Now the p-series $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} n^{-(1+\delta)}$ converges since $1+\delta > 1$. By W M-test $\sum n^{-z}$ conv. unif on B. Hence $\zeta(z)$ is analytic on A.

$$\zeta(z) = \left(\sum n^{-z}\right)' = \sum (n^{-z})' = \sum (\log n)\, n^{-z}.$$

also conv. by Analytic Conv. Thm.