3.3 Laurent Series and the Classification of Singularities

Thm 3.3.1 (Laurent Expansion Thm)
Let A be the annulus $\{z \in \mathbb{C} \mid r_1 < |z - z_0| < r_2\}$,
for some fixed $z_0 \in \mathbb{C}$ and $0 \le r_1 < r_2 \le \infty$. (often $r_1 = 0$)
Let f be analytic on A. Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

where both series converge absolutly on A and uniformly on any closed annulus within A having center z_0, where formula for a_n and b_n given below.
Let γ be a circle with the center z_0 lying inside A.
Then

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz, \quad n = 0, 1, 2, 3, 4, \ldots$$

$$b_n = \frac{1}{2\pi i} \int_\gamma f(z)\,(z - z_0)^{n-1}\, dz, \quad n = 1, 2, 3, \ldots$$

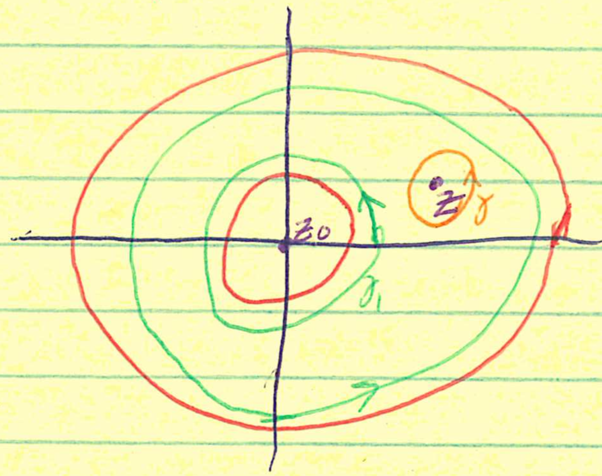
This is call the Laurent Expansion of $f(z)$ and the a_n and b_n are unique (but they depend on A, not just $f(z)$).

Examples $e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \frac{1}{4!\,z^4} + \cdots$

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$$

Thus, $\int_{|z|=1} e^{\frac{1}{z}}\, dz = 2\pi i$, $\int_{|z|=1} \frac{\sin z}{z}\, dz = 0$.

Partial Pf Cauchy's Integral Formula on an annulus A takes the following form. Let $z \in A$, z_0 is center of A. Let γ_1 and γ_2 be circles in A with center z_0 be s.t. that z is inside the disk with bdy γ_2 and outside the disk with bdy γ_1. Orient ~~each~~, γ_1, γ_2 ccw.

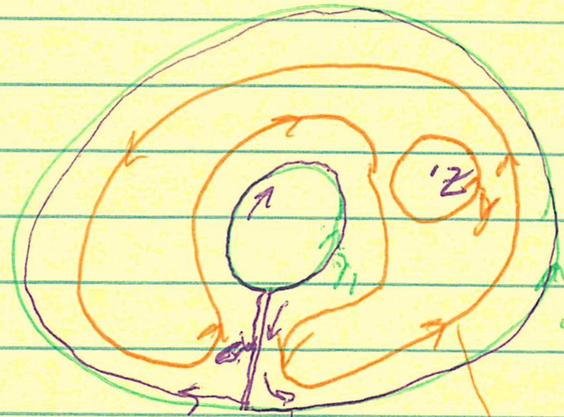


Then
$$f(z) = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{w-z}\,dw.$$

The proof of this is Exercise 5, ~~which~~ which is done in the back of the book, but I'll outline it here.

Let γ be a small circle with center z that is inside the annulus bdd by γ_1 and γ_2. Orient γ ccw. Then we know
$$f(w) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(w)}{w-z}\,dw.$$

The pictures on the next page show why $\int_\gamma = \int_{\gamma_2} - \int_{\gamma_1}$.



γ' homotopic to γ.

γ'' homotopic to γ'

and homotopic to $\gamma_2 - \sigma - \gamma_1 + \sigma$
$= \gamma_2 - \gamma_1$.

As in the proof of Taylor's Thm we use

$$\frac{1}{w-z} = \frac{1}{w-z_0} \cdot \frac{1}{1-\left(\frac{z-z_0}{w-z_0}\right)} = \frac{1}{w-z_0}\left(\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n\right)$$

The proof relies switch sums and integrals. We will address the justifications for doing so after doing this.

$$\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(w)}{w-z}\,dw = \sum_{n=0}^{\infty}\left[\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(w)}{(w-z_0)^{n+1}}\,dw\right](z-z_0)^n$$

$$= \sum_{n=0}^{\infty} a_n (z-z_0)^n.$$

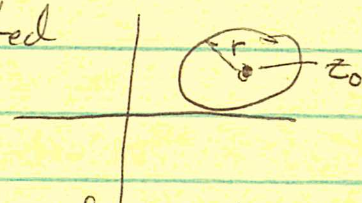
The proof that $\frac{-1}{2\pi i}\int_{\gamma_1}\frac{f(w)}{w-z}\,dw = \sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}$ is similar.

In each case there ~~are~~ is a need to justify the interchanging of the infinite sum and the integral. The first uses Prop 3.1.9 (pg 191-192) in conjunction with Exercise 21 in 3.2 (pg 221), whose solution is in the back of the book. The second, the b_n's, uses the Abel-Weierstrass Lemma (3.2.2, pgs 204-205) and Exersice 15 in 3.3 (pg 236), which is done in the back of the book.

It is only left to show uniqueness of the a_n's and b_n's. See textbook. See also the Example at the bottom of pg 223 and top of 224. $\boxed{///}$

Singularities and Poles.

Suppose $f(z)$ is undefined at z_0 but is analytic on some deleted nbhd: $\{z \in \mathbb{C} \mid 0 < |z - z_0| < r\}$. Then we say f has an isolated singularity at z_0.



Under these considitions we can find a Laurent Series for f on $\{z \in \mathbb{C} \mid 0 < |z - z_0| < r\}$,

$$f(z) = \cdots + \frac{b_2}{(z-z_0)^2} + \frac{b_1}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

The number b_1 is called the residue of f at z_0.

$$\int_\gamma f(z)\,dz = 2\pi i \cdot I(\gamma, z_0) \cdot b_1$$

If all the b_n's are zero we say that z_0 is a removable singularity. In this case we can define

$$\tilde{f}(z) = \begin{cases} a_0 & z = z_0 \\ f(z) & 0 < |z - z_0| < r \end{cases}$$

to get an analytic function on $D(z_0, r)$.

If ~~an~~ infinitely many b_k's are not zero, then f has an essential singularity at z_0.

$e^{\frac{1}{z}}$ has an essential sing. at 0.

Poles If $\exists N$ s.t. $n > N \Rightarrow b_n = 0$ and N ~~[illegible]~~ is the largest such integer, then we say f has a pole of order N at z_0.

A first order pole is also called a simple pole.

Def If f is analytic on a open connected set A, except for ~~[illegible]~~ poles in A (by def. poles are isolated), then f is said to meromorphic on A. If we say "f is a meromorphic function" we mean f is meromorphic on $\mathbb{C}$.

Ex $$\frac{\cos z + z e^z}{(z-7i)^3(z^7+52)}$$ is a meromorphic function.

Ex $$\frac{e^z}{1-|z|}$$ has singularities that are not isolated.

Ex (From 3.3.9 & 3.3.10) Find the order of these poles.

$\frac{\sin z}{z}$ at $z=0$. $\quad \frac{1}{z}\left(\cancel{z} - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) = \cancel{1} - \frac{z^2}{3!} + \cdots$ removable

$\frac{\cos z}{z^2}$ at $z=0$ $\quad \frac{1}{z^2} - \frac{1}{2!} + \frac{z^2}{4!} \cdots$ order $= 2$

$\frac{e^z - 1}{z^2}$ at $z=0$ $\quad \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} \cdots$ order $= 1$, simple.

Prop 3.3.4 Let f be analytic on A (open, conn) except for an isolated sing. at z_0.

1. The following are equivalent.
   - ⓪ z_0 is a removable sing.
   - ⓐ f is bdd on a deleted nbhd of z_0
   - ⓑ $\lim_{z \to z_0} f(z)$ exists (is finite)
   - ⓒ $\lim (z - z_0) f(z) = 0$.

2. z_0 is a simple pole $\iff$ $\lim_{z \to z_0} (z - z_0) f(z) \neq 0$ (but exists)
   In fact $\searrow = b_1$.

3. The following are equivalent.
   - ⓪ z_0 is a pole of order $\leq k$ ($k = 0 \iff$ removable)
   - ⓐ $f(z)(z - z_0)^k$ is bdd on a deleted nbhd of z_0
   - ⓑ $\lim (z - z_0)^{k+1} f(z) = 0$
   - ⓒ $\lim (z - z_0)^k f(z)$ exists.

4. z_0 is a pole of order $k \geq 1$ $\iff$ $f(z)(z - z_0)^k$ ~~is analytic on~~ has a removable sing. at z_0.

Pf: See textbook

Prop 3.3.5 Let $h(z) = \frac{f(z)}{g(z)}$ where f & g are analytic on a deleted nbhd of z_0 with (isolated) zeros of order n and k, resp.
Then

1. $k > n \Rightarrow$ h has a pole of order $k-n$ at z_0.
2. $k = n \Rightarrow$ h has a removable sing. at z_0
3. $k < n \Rightarrow$ h has a removable sing at z_0 and setting $h(z_0) = 0$ ~~remove~~ make h analytic ~~at z_0~~ on a nbhd of z_0

Pf See textbook.

Ex $\dfrac{e^z}{(z+2i)^{12}(z^2+1)^3}$ Has three poles. Order 12 at $-2i$, order 3 at $\pm i$.

Behavior near Essential Singularities.

Ex Let $f: \mathbb{R}-\{0\} \to \mathbb{R}$ be $f(x) = \frac{1}{x}\sin\frac{1}{x}$.
$\forall \varepsilon > 0$
$\forall y \in \mathbb{R}$ $\exists$ infinitely many $x \in (-\varepsilon, \varepsilon)$ with
$f(x) = y$.

For complex valued functions the behavior near essential singularities is a lot like this example.

Ex Let $f(z) = e^{\frac{1}{z}}$, $z \in \mathbb{C} - \{0\}$. Then $z_0 = 0$ is an ess. sing.
We know $e^{\frac{1}{z}}$ is never zero. But, it can be shown that
$\forall \varepsilon > 0$, $\forall w \in \mathbb{C} - \{0\}$, $\exists$ infinitely many $z \in \{ 0 < |z| < \varepsilon \}$
s.t. $f(z) = w$.

Thm This is true in general. It is called <u>Picard's Thm</u>:

Suppose $f: \overset{\text{open}}{A} \to \mathbb{C}$ has an ess. sing. at $z_0 \in A$.
Suppose $\varepsilon > 0$ and $\{ z \in \mathbb{C} \mid 0 < |z - z_0| < \varepsilon \} \subset A$.
Then $\forall w \in \mathbb{C}$, with one possible exception, $\exists$ infinitely many z with $0 < |z - z_0| < \varepsilon$ and $f(z) = w$.

The proof is outside the scope of this course.
There is a weaker version call the Casorati-Weierstrass Thm whose proof you can read in the textbook.

Ex Find and classify the sing. of $\dfrac{1-\cos(z^3)}{(e^z-1)^{15}}$.

Sol. $1-\cos(z^3) = 1-\left(1-\frac{z^6}{2}+\frac{z^{12}}{4!}-\frac{z^{18}}{6!}+\cdots\right)$

$= z^6\left(\frac{1}{2}-\frac{z^6}{4!}+\frac{z^{12}}{6!}-\cdots\right)$ zero of order 6 at $z=0$.

$(e^z-1)^{15} = \left(z+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\cdots\right)^{15} = z^{15}\left(1+\frac{z}{2!}+\frac{z^2}{3!}+\cdots\right)^{15}$

zero of order 15 at $z=0$.

Hence pole of order $15-6=9$ at $z=0$.

Are there other poles? Yes, $z=2\pi n i$ will have

$e^{2\pi n i}=1$. Since e^z is period, these have the same order, this $\frac{1}{(e^z-1)^{15}}$ will have poles of order 15 at $z=2\pi n i$.

You can check $1-\cos((2\pi n i)^3)$ is never zero. $n\neq 0$.
(This is not trivial.)

So $\dfrac{1-\cos(z^3)}{(e^z-1)^{15}}$ has poles of order 15 at $z=2\pi n i$, $n\neq 0$.

Ex 18 in 3.3. Let γ be $|z|=1$. Compute $\int_\gamma z^n e^{\frac{1}{z}} dz$.

Sol. $\displaystyle\int_\gamma z^n \left(1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \cdots + \frac{1}{(n+1)!\,z^{n+1}} + \cdots \right) dz$

$$= \int_\gamma \left(\cdots\cdots + \frac{1}{(n+1)!\,z} + \cdots \right) dz = \frac{2\pi i}{(n+1)!}.$$

Ex (Example 3.3.8 (b)) Find Laurent series of $\frac{z}{z^2+1}$, $z_0 = i$.

$$f(z) = \frac{z}{z^2+1} = \frac{\frac{1}{2}}{z-i} + \frac{\frac{1}{2}}{z+i}$$

$$\frac{1}{z+i} = \frac{1}{2i + (z-i)} = \frac{1}{2i}\,\frac{1}{1-\left(\frac{i-z}{2i}\right)} = \frac{1}{2i}\sum_{n=0}^{\infty} (-1)^n \left(\frac{z-i}{2i}\right)^n$$

$$= \frac{-i}{2} \sum \left(\frac{-1}{i}\right)^n \frac{1}{2^n} (z-i)^n = \frac{-i}{2} \sum_{n=0}^{\infty} \left(\frac{i}{2}\right)^n (z-i)^n.$$

Thus,

$$f(z) = \frac{1}{2}(z-i)^{-1} - \sum_{n=0}^{\infty} \left(\frac{i}{2}\right)^{n+1} (z-i)^n$$

Res $= \frac{1}{2}$. Converges on $0 < |z-i| < 2$.

Theory and Problems of Complex Variables
— M.R. Spiegel, 1964

Hence from Property 5, Page 93, we have,

$$|V_n| = \frac{1}{2\pi}\left|\oint_{C_2}\left(\frac{w-a}{z-a}\right)^n \frac{f(w)}{z-w}\,dw\right| \leqq \frac{1}{2\pi}\frac{\kappa^n M}{|z-a|-r_2}2\pi r_2 = \frac{\kappa^n M r_2}{|z-a|-r_2}$$

Then $\lim_{n\to\infty} V_n = 0$ and the proof is complete.

26. Find Laurent series about the indicated singularity for each of the following functions. Name the singularity in each case and give the region of convergence of each series.

(a) $\dfrac{e^{2z}}{(z-1)^3}$; $z=1$. Let $z-1=u$. Then $z=1+u$ and

$$\frac{e^{2z}}{(z-1)^3} = \frac{e^{2+2u}}{u^3} = \frac{e^2}{u^3}\cdot e^{2u} = \frac{e^2}{u^3}\left\{1+2u+\frac{(2u)^2}{2!}+\frac{(2u)^3}{3!}+\frac{(2u)^4}{4!}+\cdots\right\}$$

$$= \frac{e^2}{(z-1)^3}+\frac{2e^2}{(z-1)^2}+\frac{2e^2}{z-1}+\frac{4e^2}{3}+\frac{2e^2}{3}(z-1)+\cdots$$

$z=1$ is a *pole of order 3*, or *triple pole*.

The series converges for all values of $z \neq 1$.

(b) $(z-3)\sin\dfrac{1}{z+2}$; $z=-2$. Let $z+2=u$ or $z=u-2$. Then

$$(z-3)\sin\frac{1}{z+2} = (u-5)\sin\frac{1}{u} = (u-5)\left\{\frac{1}{u}-\frac{1}{3!\,u^3}+\frac{1}{5!\,u^5}-\cdots\right\}$$

$$= 1-\frac{5}{u}-\frac{1}{3!\,u^2}+\frac{5}{3!\,u^3}+\frac{1}{5!\,u^4}-\cdots$$

$$= 1-\frac{5}{z+2}-\frac{1}{6(z+2)^2}+\frac{5}{6(z+2)^3}+\frac{1}{120(z+2)^4}-\cdots$$

$z=-2$ is an *essential singularity*.

The series converges for all values of $z \neq -2$.

(c) $\dfrac{z-\sin z}{z^3}$; $z=0$.

$$\frac{z-\sin z}{z^3} = \frac{1}{z^3}\left\{z-\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\cdots\right)\right\}$$

$$= \frac{1}{z^3}\left\{\frac{z^3}{3!}-\frac{z^5}{5!}+\frac{z^7}{7!}-\cdots\right\} = \frac{1}{3!}-\frac{z^2}{5!}+\frac{z^4}{7!}-\cdots$$

$z=0$ is a *removable singularity*.

The series converges for all values of z.

(d) $\dfrac{z}{(z+1)(z+2)}$; $z=-2$. Let $z+2=u$. Then

$$\frac{z}{(z+1)(z+2)} = \frac{u-2}{(u-1)u} = \frac{2-u}{u}\cdot\frac{1}{1-u} = \frac{2-u}{u}(1+u+u^2+u^3+\cdots)$$

$$= \frac{2}{u}+1+u+u^2+\cdots = \frac{2}{z+2}+1+(z+2)+(z+2)^2+\cdots$$

$z=-2$ is a pole of order 1, or simple pole.

The series converges for all values of z such that $0<|z+2|<1$.

(e) $\dfrac{1}{z^2(z-3)^2}$; $z=3$. Let $z-3=u$. Then by the binomial theorem,

$$\begin{aligned}\frac{1}{z^2(z-3)^2} &= \frac{1}{u^2(3+u)^2} = \frac{1}{9u^2(1+u/3)^2}\\ &= \frac{1}{9u^2}\left\{1 + (-2)\left(\frac{u}{3}\right) + \frac{(-2)(-3)}{2!}\left(\frac{u}{3}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(\frac{u}{3}\right)^3 + \cdots\right\}\\ &= \frac{1}{9u^2} - \frac{2}{27u} + \frac{1}{27} - \frac{4}{243}u + \cdots\\ &= \frac{1}{9(z-3)^2} - \frac{2}{27(z-3)} + \frac{1}{27} - \frac{4(z-3)}{243} + \cdots\end{aligned}$$

$z = 3$ is a *pole of order 2* or *double pole*.

The series converges for all values of z such that $0 < |z-3| < 3$.

27. Expand $f(z) = \dfrac{1}{(z+1)(z+3)}$ in a Laurent series valid for (a) $1 < |z| < 3$, (b) $|z| > 3$, (c) $0 < |z+1| < 2$, (d) $|z| < 1$.

(a) Resolving into partial fractions, $\dfrac{1}{(z+1)(z+3)} = \dfrac{1}{2}\left(\dfrac{1}{z+1}\right) - \dfrac{1}{2}\left(\dfrac{1}{z+3}\right)$.

If $|z| > 1$,

$$\frac{1}{2(z+1)} = \frac{1}{2z(1+1/z)} = \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right) = \frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \cdots$$

If $|z| < 3$,

$$\frac{1}{2(z+3)} = \frac{1}{6(1+z/3)} = \frac{1}{6}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right) = \frac{1}{6} - \frac{z}{18} + \frac{z^2}{54} - \frac{z^3}{162} + \cdots$$

Then the required Laurent expansion valid for both $|z| > 1$ and $|z| < 3$, i.e. $1 < |z| < 3$, is

$$\cdots - \frac{1}{2z^4} + \frac{1}{2z^3} - \frac{1}{2z^2} + \frac{1}{2z} - \frac{1}{6} + \frac{z}{18} - \frac{z^2}{54} + \frac{z^3}{162} - \cdots$$

(b) If $|z| > 1$, we have as in part (a),

$$\frac{1}{2(z+1)} = \frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \cdots$$

If $|z| > 3$,

$$\frac{1}{2(z+3)} = \frac{1}{2z(1+3/z)} = \frac{1}{2z}\left(1 - \frac{3}{z} + \frac{9}{z^2} - \frac{27}{z^3} + \cdots\right) = \frac{1}{2z} - \frac{3}{2z^2} + \frac{9}{2z^3} - \frac{27}{2z^4} + \cdots$$

Then the required Laurent expansion valid for both $|z| > 1$ and $|z| > 3$, i.e. $|z| > 3$, is by subtraction

$$\frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \frac{40}{z^5} + \cdots$$

(c) Let $z + 1 = u$. Then

$$\begin{aligned}\frac{1}{(z+1)(z+3)} &= \frac{1}{u(u+2)} = \frac{1}{2u(1+u/2)} = \frac{1}{2u}\left(1 - \frac{u}{2} + \frac{u^2}{4} - \frac{u^3}{8} + \cdots\right)\\ &= \frac{1}{2(z+1)} - \frac{1}{4} + \frac{1}{8}(z+1) - \frac{1}{16}(z+1)^2 + \cdots\end{aligned}$$

valid for $|u| < 2,\ u \neq 0$ or $0 < |z+1| < 2$.

(d) If $|z| < 1$,

$$\frac{1}{2(z+1)} = \frac{1}{2(1+z)} = \tfrac{1}{2}(1 - z + z^2 - z^3 + \cdots) = \tfrac{1}{2} - \tfrac{1}{2}z + \tfrac{1}{2}z^2 - \tfrac{1}{2}z^3 + \cdots$$

If $|z| < 3$, we have by part (a),

$$\frac{1}{2(z+3)} = \frac{1}{6} - \frac{z}{18} + \frac{z^2}{54} - \frac{z^3}{162} + \cdots$$

Then the required Laurent expansion, valid for both $|z| < 1$ and $|z| < 3$, i.e. $|z| < 1$, is by subtraction

$$\frac{1}{3} - \frac{4}{9}z + \frac{13}{27}z^2 - \frac{40}{81}z^3 + \cdots$$

This is a *Taylor series*.