4.1 Finding Residues

See Table 4.1.1. (pg 250). You can use it during tests. We will derive a few of the formulas in it and then do examples.

<u>Prop 4.1.1</u> If $g(z)$ and $h(z)$ analytic and have zeros of the same order (k) at z_0, then $f(z) = g(z)/h(z)$ has a removable sing. at z_0 and hence $\mathrm{Res}(\frac{g}{h}, z_0) = 0$.

<u>Pf</u> By Taylor's Thm we may write $g(z) = (z-z_0)^k \tilde{g}(z)$ and $h(z) = (z-z_0)^k \hat{h}(z)$ where $\tilde{g}(z)$ and $\hat{h}(z)$ are analytic and nonzero at z_0. Thus,

$$\lim_{z \to z_0} \frac{g(z)}{h(z)} = \frac{\tilde{g}(z_0)}{\hat{h}(z_0)} \quad \text{exists.}$$

By Prop 3.3.4 the sing. is removable and Res $= 0$.

<u>Ex</u> $\mathrm{Res}\left(\frac{\sin z}{z}, 0\right) = 0.$

Ex $\mathrm{Res}\left(\frac{e^z - 1}{z}, 0\right) = 0.$

Prop 4.1.2 Let $g(z)$ and $h(z)$ be analytic at z_0 with $g(z_0)\neq 0$, $h(z_0)=0$, but $h'(z_0)\neq 0$. Then $\frac{g}{h}$ has a simple pole at z_0 and

$$\operatorname{Res}\left(\frac{g}{h}, z_0\right) = \frac{g(z_0)}{h'(z_0)}$$

Pf Use Taylor series. $g(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$

$h(z) = c_1(z-z_0) + c_2(z-z_0)^2 + \cdots$

Then $\displaystyle\lim_{z\to z_0} \frac{(z-z_0)\,g(z)}{h(z)} = \frac{a_0}{c_1} \neq 0.$

By Prop. 3.3.4 we have a similar pole.

Notice that $\frac{a_0}{c_1} = \frac{g(z_0)}{h'(z_0)}$, so this is the residue. $\square$

Ex Let $f(z) = \dfrac{z e^z}{(z-1)\sin(z)}$. Find Residues at each pole.

Sol Removable sing at $z=0$. Hence $\operatorname{Res}(f,0)=0$.

Simple pole at $z=1$.

① $\displaystyle\operatorname{Res}(f,1) = \lim_{z\to \cancel{z_0}} \frac{(z-1)ze^z}{(z-1)\sin z} = \frac{e}{\sin(1)}.$

② Let $g(z) = ze^z$. $g(1) = e \neq 0$. Let $h = (z-1)\sin z$. $h(1)=0$

$h'(z) = \sin z + (z-1)\cos z$. $h'(1) = \sin(1)$.

$$\operatorname{Res}(f,0) = \frac{e}{\sin(1)}.$$

Ex $f(z) = \dfrac{\cos z\, e^z}{\sin(\sin(z))}$. Find $\text{Res}(f, 0)$.

Sol. Let $g(z) = \cos z\, e^z$. $g(0) = 1$

Let $h(z) = \sin(\sin(z))$. $h(0) = 0$

$h'(z) = \cos(\sin(z)) \cdot \cos(z)$ $h'(0) = 1$.

Thus, $\text{Res}(f, 0) = \dfrac{g(0)}{h'(0)} = 1$.

Q: Are there other poles of $f(z)$?

Prop 4.13. Let $f(z) = \frac{g(z)}{h(z)}$ where $g(z)$ has a pole at z_0 of order k and $h(z)$ has a pole at z_0 of order $k+1$, both are analytic on same ~~d~~ deleted disk center z_0. Then f has a simple pole at z_0 and

$$\operatorname{Res}\left(\frac{g}{h}, z_0\right) = (k+1)\,\frac{g^{(k)}(z_0)}{h^{(k+1)}(z_0)}$$

Outline of Pf Use Taylor series to get

$$\frac{(\cancel{z-z_0})\,g(z)}{h(z)} = \frac{(z-z_0)^k/k!\; g^{(k)}(z_0) + (z-z_0)^{k+1}\tilde{g}(z)}{\frac{(z-z_0)^{k+1}}{(k+1)!}\, h^{(k+1)}(z_0) + (z-z_0)^{k+2}\,\tilde{h}(z)}$$

$$\frac{(z-z_0)\,g(z)}{h(z)} = \frac{\frac{g^{(k)}(z_0)}{k!} + (z-z_0)\,\tilde{g}(z)}{\frac{h^{(k+1)}(z_0)}{(k+1)!} + (z-z_0)^{k+1}\,\tilde{h}(z)}$$

$$\lim_{z \to z_0} \frac{(z-z_0)\,g(z)}{h(z)} = (k+1)\,\frac{g^{(k)}(z_0)}{h^{(k+1)}(z_0)} = \operatorname{Res}(f, z_0)$$

Prop 4.1.4 Let $g(z)$ and $h(z)$ be analytic ~~at z_0~~ on a nbhd of z_0.

If $g(z_0) \neq 0$, $h(z_0) = 0$, $h'(z_0) = 0$, $h''(z_0) = 0$,

then g/h has a second-order pole at z_0 and

$$\operatorname{Res}\left(\tfrac{g}{h}, z_0\right) = 2\,\frac{g'(z_0)}{h''(z_0)} - \frac{2}{3}\,\frac{g(z_0)\,h'''(z_0)}{[h''(z_0)]^2}.$$

Pf It is clear we have a second-order pole at z_0.

Thus, the Laurent series for g/h at z_0 is

$$\frac{g(z)}{h(z)} = \frac{b_2}{(z-z_0)^2} + \frac{b_1}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

Find b_1 !

But $g(z)$ and $h(z)$ have Taylor series at z_0.

$$g(z) = g(z_0) + g'(z_0)(z-z_0) + \frac{g''(z_0)}{2}(z-z_0)^2 + \frac{g'''(z_0)}{3!}(z-z_0)^3 + \cdots$$

$$h(z) = 0 + 0 + \frac{h''(z_0)}{2}(z-z_0)^2 + \frac{h'''(z_0)}{3!}(z-z_0)^3 + \cdots$$

Now $g(z) = h(z) \cdot \dfrac{g(z)}{h(z)}$

$$= \frac{b_2}{2}h''(z_0) + \left[\frac{b_2 h'''(z_0)}{3!} + \frac{b_1 h''(z_0)}{2}\right](z-z_0) + \cdots$$

$$\Rightarrow g(z_0) = \frac{b_2 h''(z_0)}{2} \quad \text{and} \quad g'(z_0) = \left[\frac{b_2 h'''(z_0)}{3!} + \frac{b_1 h''(z_0)}{2}\right]$$

$$\Rightarrow b_2 = \frac{2g(z_0)}{h''(z_0)} \Rightarrow g'(z_0) = \left[\frac{2g(z_0)h'''(z_0)}{3!\,h''(z_0)} + \frac{b_1 h''(z_0)}{2}\right]$$

$$\Rightarrow b_1 = \left(g'(z_0) - \frac{g(z_0)h'''(z_0)}{3h''(z_0)}\right)\frac{2}{h''(z_0)} = \frac{2g'(z_0)}{h''(z_0)} - \frac{2}{3}\frac{g(z_0)h'''(z_0)}{(h(z_0))^2}. \checkmark$$

Ex Find the residue of $f(z) = \dfrac{e^z}{\sin^2 z}$ at $z_0 = 0$.

Sol Let $g(z) = e^z$ and $h(z) = \sin^2 z$. Looks like second-order pole at $z=0$.

$g(0) = 1 \neq 0$ $\quad h(0) = 0$

$h'(z) = 2\sin z \cos z = \sin(2z)$ $\quad h'(0) = 0$

$h''(z) = 2\cos(2z)$ $\quad h''(0) = 2 \neq 0$

Hence formula #6 in Table 4.1.1 applies.

$h'''(z) = -4\sin(2z)$ $\quad h'''(0) = 0$ $\quad g'(z) = e^z,\ g'(0) = 1.$

$$\operatorname{Res}(f, 0) = 2\frac{g'(0)}{h''(0)} - \frac{2}{3}\frac{g(0)h'''(0)}{[h''(0)]^2} = 2 \cdot \frac{1}{2} - \frac{2}{3} \cdot \frac{1 \cdot 0}{2^2} = 1.$$

Note: In fact there are poles of order 2 at $z_0 = n\pi$, $\forall n \in \mathbb{Z}$. You can show that $R(f, n\pi) = e^{n\pi}$.

Ex Let $f(z) = \dfrac{\cos z}{z \sin z}$. Find residue at $z_0 = n\pi$, $\forall n \in \mathbb{Z}$.

Solution For $n \neq 0$, the poles at $z_0 = n\pi$ are simple. We do this case first. Let $g(z) = \cos z$ and $h(z) = z\sin z$. $g(n\pi) = (-1)^n \neq 0$. $h(0) = 0$. $h'(z) = \sin z + z\cos z$. $h'(n\pi) = n\pi(-1)^n$

$$\operatorname{Res}(f, n\pi) = \frac{g(n\pi)}{h'(n\pi)} = \frac{1}{n\pi}.$$

At $z=0$ we have a double-pole.

$g'(z) = \sin z \qquad g'(0) = 0.$

$h'(0) = 0$

$h''(z) = 2\cos z + z\sin z \qquad h''(0) = 2.$

$h'''(z) = -2\sin z + \sin z + z\cos z \qquad h'''(0) = 0.$

Thus,

$$\mathrm{Res}(f, 0) = 0, \text{ by formula 6.}$$

Ex Find residue of $f(z) = \dfrac{ze^z}{\sin^3 z}$ at $z=0$.

Sol $g(z) = ze^z. \quad g(0) = 0. \qquad g'(z) = e^z + ze^z. \quad g'(0) = 1.$

$h(z) = \sin^3 z \qquad h(0) = 0$

$h'(z) = 3\sin^2 z\cos z. \qquad h'(0) = 0$

$h''(z) = 6\sin z\cos^2 z - 3\sin^3 z \qquad h''(0) = 0$

$h'''(z) = 6\cos^3 z - 12\sin^2 z\cos z - 9\sin^2 z\cos z$

$\qquad\qquad -18 s^2 c$

$h'''(0) = 6$

Use formula 8. Need: $h''''(z) = -18\cos^2 z\sin z$

$\qquad - 42 \sin z\cos^2 z$

$\qquad + 18\sin^3 z$

$g''(z) = 2e^z + ze^z$

$g''(0) = 2 \qquad\qquad h''''(0) = 0 \quad$ yea!

$$\mathrm{Res}(f, 0) = 3\frac{g''(0)}{h'''(0)} - 0 = 3 \cdot \frac{2}{6} = 1.$$