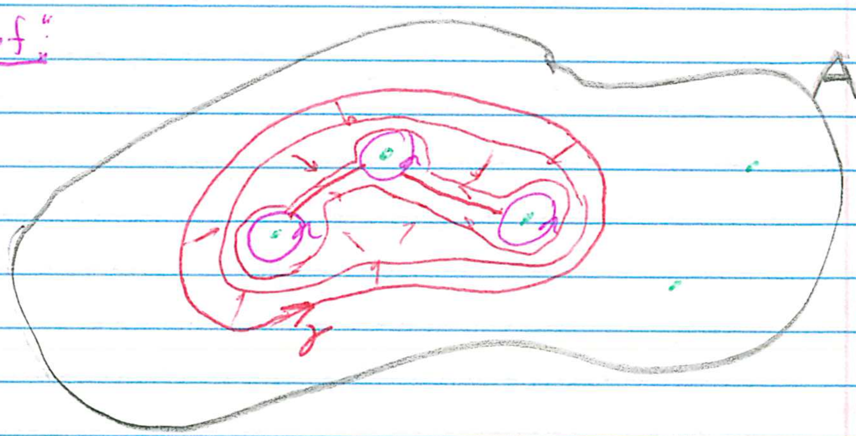
4.2 The Residue Thm

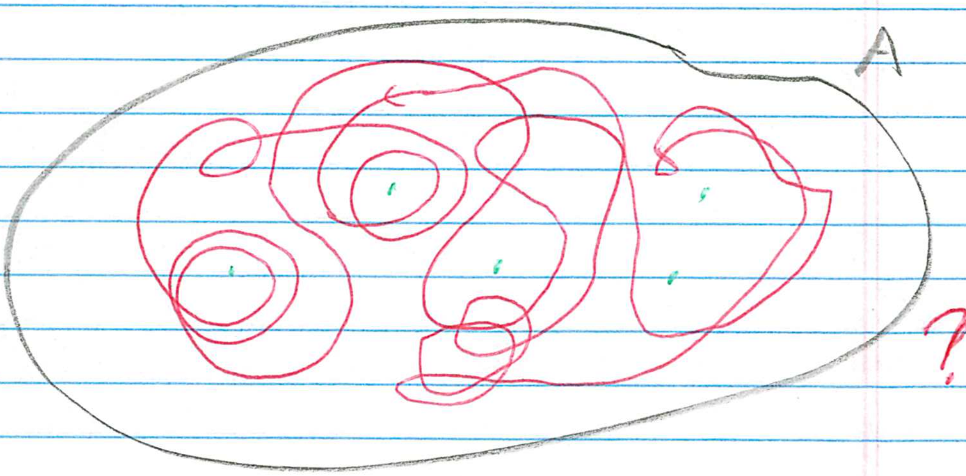
Thm (4.2.1) Let $f(z)$ be a function which is analytic on an open conn'd set $A \subset \mathbb{C}$, except at a finite subset of isolated singularities $\{z_1, z_2, \ldots, z_n\}$. Let γ be a closed curve in A homotopic to a point in A with no z_i lying on γ. Then

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{i=1}^{n} \operatorname{Res}(f, z_i) \cdot I(\gamma, z_i).$$

Intuitive "Proof":



But would this "work" for ....



Outline of formal Proof. For each $i = 1, \dots, n$, there is a Laurent series for $f(z)$ on some deleted nbhd $D(z_i, r_i) - \{z_i\}$ $r_i > 0$,

$$f(z) = \sum_{p=0}^{\infty} a_{ip} (z - z_i)^p + \sum_{m=1}^{\infty} \frac{b_{im}}{(z - z_i)^m}, \quad i = 1 \dots n.$$

Let $$S_i(z) = \sum_{m=1}^{\infty} \frac{b_{im}}{(z - z_i)^m}, \quad i = 1 \dots n.$$

We know $S_i(z)$ converges inside $D(z_i, r_i) - \{z_i\}$. But it also converges for $|z| \geq r_i$ by the Comparison Test. In fact the convergence is uniform (why?), thus $S_i(z)$ is an analytic function on $\mathbb{C} - \{z_i\}$, for each $i = 1 \dots n$.

Define $g(z) = f(z) - \sum_{i=1}^{n} S_i(z)$.

Then $g(z)$ is analytic on $A - \{z_1, \dots z_n\}$. We will show that the points, $z_1, \dots z_n$, are removable sing. for $g(z)$.

Fix at index i. On $D(z_i, r_i) - \{z_i\}$ we have

$$f(z) = \sum_{p=0}^{\infty} a_{ip} (z - z_i)^p + S_i(z).$$

Thus,

$$g(z) = \left(\sum_{p=0}^{\infty} a_{ip} (z - z_i)^p + S_i(z) \right) - \sum_{j=1}^{n} S_j(z)$$

$$= \sum_{p=0}^{\infty} a_{ip} (z - z_i)^p - \sum_{\substack{j=1 \\ j \neq i}}^{n} S_j(z)$$

Thus $\lim_{z \to z_i} g(z) = a_{i0} - \sum_{\substack{j=1 \\ j \neq i}}^{n} S_j(z_i)$; so it exists.

By Prop. 3.3.4 (b), z_i is a removable sing of $g(z)$. This works for $i = 1 \dots n$.

It follows that $\int_\gamma g(z)\,dz = 0$. Thus, $\int_\gamma f(z)\,dz = \sum_{i=1}^{n} \int_\gamma S_i(z)\,dz$.

Next we consider the integrals $\int_\gamma S_i(z)\,dz$. But these are easy.

$$\int_\gamma S_i(z)\,dz = \int_\gamma \sum_{m=1}^{\infty} \frac{b_{im}}{(z-z_i)^m}\,dz = \int_\gamma \frac{b_{i1}}{(z-z_i)}\,dz = 2\pi i\, b_{i1} \cdot I(\gamma, z_i)$$

$$= 2\pi i \operatorname{Res}(f, z_i) \cdot I(\gamma, z_i)$$

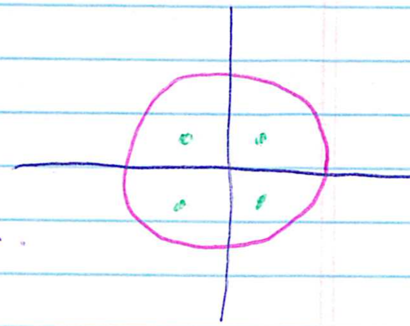
Thus,

$$\int_\gamma f(z)\,dz = \sum_{i=1}^{n} 2\pi i \operatorname{Res}(f, z_i)\, I(\gamma, z_i). \qquad \boxtimes$$

Next we do an example.

Example Compute $\int_\gamma \frac{1}{z^4+1}\,dz$, where $\gamma = 2e^{it}$, $0 \le t \le 2\pi$.

Solution We know $z^4 = -1$ for $z = e^{\frac{\pi}{4}i}, e^{\frac{3\pi}{4}i}, e^{\frac{5\pi}{4}i}$ and $e^{\frac{7\pi}{4}i}$, and these are inside γ.



We could use a partial faction expansion

$$\frac{1}{z^4+1} = \frac{A}{z-e^{\pi/4 i}} + \frac{B}{z-e^{\frac{3\pi}{4}i}} + \frac{C}{z-e^{\frac{5\pi}{4}i}} + \frac{D}{z-e^{\frac{7\pi}{4}i}}.$$

We just have to find A, B, C, D. Or, we could just find the residue at these poles.

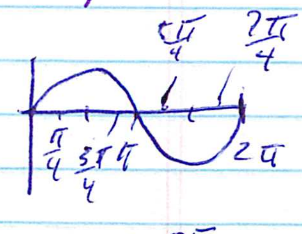
$$\text{Res}\left(\frac{1}{z^4+1}, w\right) = \frac{1}{4w^3}.$$

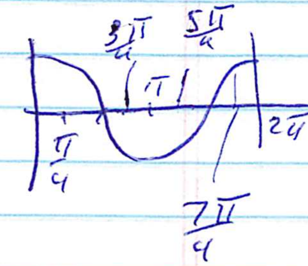
$$\frac{1}{4e^{\frac{3\pi}{4}i}} = \frac{1}{4}e^{-\frac{3\pi}{4}i}, \quad \frac{1}{4^{\frac{9\pi}{4}i}} = \frac{1}{4e^{\pi/4 i}} = \frac{1}{4}e^{-\frac{\pi}{4}i},$$

$$\frac{1}{4e^{\frac{15\pi}{4}i}} = \frac{1}{4e^{\frac{7\pi}{4}i}} = \frac{1}{4}e^{-\frac{7\pi}{4}i} \text{ and } \frac{1}{4e^{21\pi/4 i}} = \frac{1}{4}e^{-\frac{5\pi}{4}i}$$

Thus, $\int_\gamma \frac{1}{z^4+1}\,dz = \frac{2\pi i}{4}\left(e^{-\frac{3\pi}{4}i} + e^{-\frac{\pi}{4}i} + e^{-\frac{7\pi}{4}i} + e^{-\frac{5\pi}{4}i}\right)$

$$= \frac{2\pi i}{4}\Big(\cos\left(\tfrac{3\pi}{4}\right) - i\sin\left(\tfrac{3\pi}{4}\right)$$
$$+ \cos\left(\tfrac{\pi}{4}\right) - i\sin\left(\tfrac{\pi}{4}\right)$$
$$+ \cos\left(\tfrac{7\pi}{4}\right) - i\sin\left(\tfrac{7\pi}{4}\right)$$
$$+ \cos\left(\tfrac{5\pi}{4}\right) - i\sin\left(\tfrac{5\pi}{4}\right)\Big) = 0$$





Residues at ∞.

Def (4.2.2). Let f be a give complex valued function with domain in $\mathbb{C}$.

Let $F(z) = f\left(\frac{1}{z}\right)$.

(i) We will say f has a pole of order k at ∞, if $F(z)$ has a ~~pole~~ zero of order k at 0.

(ii) We will say f has a zero of order k at ∞, if $F(z)$ has a pole of order k at 0.

(iii) We define $\operatorname{Res}(f, \infty) = -\operatorname{Res}\left(\frac{F(z)}{z^2}, 0\right)$.

<u>Motivation</u> The $-$ sign is b/c at ∞ cw becomes ccw.

~~If $F(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ $\operatorname{Res}\left(\frac{F(z)}{z^2}, 0\right) = a_1$~~

If $f(z) = \cdots \frac{b_3}{z^3} + \frac{b_2}{z^2} + \frac{b_1}{z} + a_0 + a_1 z + a_2 z^2 \cdots$,

Then $\frac{f\left(\frac{1}{z}\right)}{z^2} = b_3 z + b_2 + \frac{b_1}{z} + \frac{a_0}{z^2} + \frac{a_1}{z^3} + \cdots$ $\quad \operatorname{Res}\left(\frac{f\left(\frac{1}{z}\right)}{z^2}, \text{"}0\text{"}\right) = b_1$

<u>Prop 4.2.4</u> Let γ be a simple closed curve in $\mathbb{C}$, ccw. Let f be analytic along γ and have finitely many iso. sing outside γ. Then

$$\int_\gamma f(z)\,dz = -2\pi i \sum \text{residues of } f \text{ outside } \gamma \text{ including residue at } \infty.$$

<u>Pf</u>: See Prop 4.2.3 and Exercise 14.

Example $\int_\gamma \frac{1}{z^4+1}\,dz \quad , \gamma = 2e^{it}, \quad 0 \le t \le 2\pi.$

Solution The only pole outside γ is at ∞.

Let $f(z) = \frac{1}{z^4+1}$. Let $F(z) = f\left(\frac{1}{z}\right) = \frac{1}{\frac{1}{z^4}+1} = \frac{z^4}{1+z^4}$

$\frac{F(z)}{z^2} = \frac{z^2}{1+z^4}, \; z \neq 0$. The sing. at $z=0$ is removable.

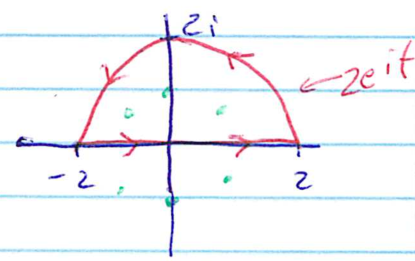
Thus, $\text{Res}\left(\frac{F(z)}{z^2}, 0\right) = 0.$

Thus $\int_\gamma \frac{1}{z^4+1}\,dz = -2\pi i \cdot 0 = 0.$

See Example 4.2.7. pages 266-8.

The next example will lead into 4.3.

Example Let $f(z) = \frac{z^2}{z^6+1}$ and $\gamma =$



~~Solution~~ Find $\int_\gamma f(z)\,dz$

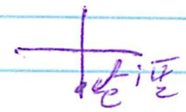
Solution The singularities of f are at $e^{i\frac{\pi}{6} + \frac{2\pi n i}{6}}$, $n = 0,1,2,3,4,5.$

All are simple poles. The first 3 are inside γ.
None are on γ.

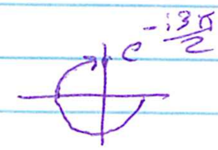
$$\int_\gamma f = 2\pi i \left[\text{Res}\left(f, e^{i\frac{\pi}{6}}\right)\cdot 1 + \text{Res}\left(f, e^{i\frac{\pi}{2}}\right)\cdot 1 + \text{Res}\left(f, e^{i\frac{5\pi}{6}}\right)\right]$$

Let $g(z) = z^2$, $h(z) = z^6 + 1$. $\text{Res}(f, z_0) = \dfrac{g(z_0)}{h'(z_0)} = \dfrac{z_0^2}{6z_0^5} = \dfrac{1}{6z_0^3}$.

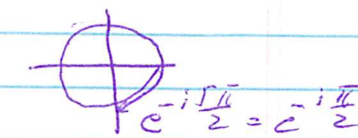
$\text{Res}(f, e^{i\frac{\pi}{6}}) = \frac{1}{6} e^{-i\frac{\pi}{2}} = \frac{-i}{6}$.



$\text{Res}(f, e^{i\frac{\pi}{2}}) = \frac{1}{6} e^{-i\frac{3\pi}{2}} = \frac{i}{6}$



$\text{Res}(f, e^{i\frac{5\pi}{6}}) = \frac{1}{6} e^{-i\frac{5\pi}{2}} = \frac{1}{6} e^{-i\frac{\pi}{2}} = \frac{-i}{6}$



$$\int_\gamma f = 2\pi i \left(\frac{-i}{6} + \frac{i}{6} + \frac{-i}{6} \right) = \frac{\pi}{3}$$