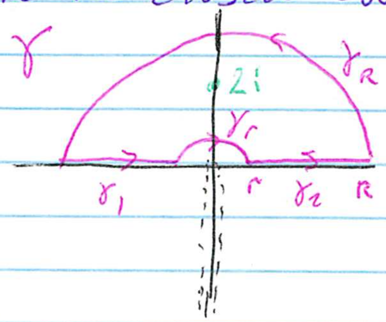
Example Show that $\int_0^\infty \frac{\ln x}{(x^2+4)^2}\,dx = \frac{\pi}{32}(\ln 2 - 1) \approx -0.03$

Solution We will make a branch cut along the neg. imaginary axis and let γ be the closed curve below.



$0 < r < 2 < R$

Let $f(z) = \frac{\ln z}{(z^2+4)^2}$. We will compute $\int_\gamma f(z)dz$.

But $\int_\gamma f = \int_{\gamma_R} f + \int_{\gamma_1} f + \int_{\gamma_r} f + \int_{\gamma_2} f$

In the limit as $r \to 0$ and $R \to 0$, $\int_{\gamma_r} f \to 0$, $\int_{\gamma_R} f \to 0$,

as we will show. The limit of $\int_{\gamma_2} f$ is $\int_0^\infty f(x)dx = I$

Less obvious, it will turn out that $\int_{\gamma_1} f \to I + \pi i \int_0^\infty \frac{1}{(x^2+4)^2}\,dx$.

This will allow us to solve for I.

Step 1 Compute $\int_\gamma f(z)dz$.

It is $= 2\pi i \operatorname{Res}(f, 2i)$. $z_0 = 2i$ is a second order pole of $f(z)$.

I will use formula 9 with $k=2$. in Table 4.1.1., but you could also use formula 6.

Let $\varphi(z) = (z-2i)^2 \frac{\ln z}{(z^2+4)^2} = \frac{\ln z}{(z+2i)^2}$

$$\mathrm{Res}(f, 2i) = \lim_{z \to 2i} \left(\frac{\ln z}{(z+2i)^2}\right)' = \lim_{z\to 2i} \left(\frac{\frac{1}{z}(z+2i)^2 - (\ln z)\, 2(z+2i)}{(z+2i)^4}\right)$$

$$= \frac{\frac{1}{2i}(4i)^2 - (\ln 2i)\, 8i}{(4i)^4} = \frac{8i - 8i(\ln 2 + i\frac{\pi}{2})}{256}$$

$$= \frac{(1-\ln 2)\, i + \frac{\pi}{2}}{32} = \frac{\pi}{64} + \frac{(1-\ln 2)}{32} i$$

Thus, $\int_\gamma f(z)\,dz = 2\pi i \left(\quad \right) = \boxed{\frac{\pi(\ln 2 - 1)}{16} + \frac{\pi^2}{32} i}$

Step 2 Show $\int_{\gamma_R} f \to 0$ as $R \to \infty$.

$$\left| \int_{\gamma_R} f(z)\,dz \right| \le \int_{\gamma_R} \frac{|\ln z|}{|z^2+4|^2} |dz| = (\star)$$

$z = Re^{it} \qquad 0 \le t \le \pi.$

$\ln z = \ln R + it$

$|\ln z| = \sqrt{(\ln R)^2 + t^2} \le \sqrt{(\ln R)^2 + \pi^2}$ on γ_R.

$\frac{1}{|z^2+4|^2} \le \frac{1}{(R^2-4)^2}$ on γ_R.

Thus, $(\star) \le \frac{\sqrt{(\ln R)^2+\pi^2}}{(R^2-4)^2} \cdot \underbrace{\pi R}_{= \text{Length of } \gamma_R.}$

Now, $\lim\limits_{R\to\infty} \dfrac{\left(\sqrt{(\ln R)^2+\pi^2}\right)\pi R}{(R^2-4)^2} = \pi \lim\limits_{R\to\infty} \dfrac{R\sqrt{(\ln R)^2+\pi^2}}{R^4-8R^2+16}\,\dfrac{\frac{1}{R^4}}{\frac{1}{R^4}}$

$= \pi \lim\limits_{R\to 0} \dfrac{\frac{1}{R^2}\sqrt{\frac{(\ln R)^2+\pi^2}{R^4}}}{1-\frac{8}{R^2}+\frac{16}{R^4}} = \pi\,\dfrac{0}{1}\lim\limits_{R\to\infty}\sqrt{\dfrac{(\ln R)^2+\pi^2}{R^4}}$

If this last limit is finite our limit is zero.

$\lim\limits_{R\to\infty} \dfrac{(\ln R)^2+\pi^2}{R^4} \underset{\frac{\infty}{\infty}\ \text{L'Hip}}{=} \lim\limits_{R\to\infty}\dfrac{2(\ln R)\cdot\frac{1}{R}}{4R^3} = \dfrac{1}{2}\lim\limits_{R\to\infty}\dfrac{\ln R}{R^4}$ $\quad \frac{\infty}{\infty}$

$\underset{\text{L'Hip.}}{=} \dfrac{1}{2}\lim\limits_{R\to\infty}\dfrac{\frac{1}{R}}{4R^3} = \dfrac{1}{8}\lim\limits_{R\to\infty}\dfrac{1}{R^4} = 0.$

Thus, the original limit is zero, i.e. $\int_{\gamma_R} f \to 0$ as $R\to\infty$.

Step 3 Show $\int_{\gamma_r} f \to 0$ as $r\to 0$.

As before $\left|\int_{\gamma_r} f\right| \le \dfrac{\sqrt{(\ln r)^2+\pi^2}}{(4-r^2)^2}\pi r = \dfrac{\pi r\sqrt{(\ln r)^2+\pi^2}}{r^4-8r^2+16}$

As $r\to 0$, $(\ln r)^2\to\infty$. The denominator goes to 16.

Thus $\lim\limits_{r\to 0}\dfrac{\pi r\sqrt{(\ln r)^2+\pi^2}}{r^4-8r^2+16} = \dfrac{\pi}{16}\lim\limits_{r\to 0} r\sqrt{(\ln r)^2+\pi^2} = (*)$

Now $r\sqrt{(\ln r)^2+\pi^2} = \sqrt{(r\ln r)^2+r^2\pi^2}.$

Clearly, $r^2\pi^2\to 0$.

But $\lim_{r \to 0} r \ln r = \lim_{r \to 0} \frac{\ln r}{\frac{1}{r}} \underset{\text{L'Hop'}}{=} \lim_{r \to 0} \frac{\frac{1}{r}}{\frac{-1}{r^2}} = -\lim_{r \to 0} r = 0.$

$\frac{-\infty}{\infty}$

Thus, limit (*) is 0, ie $\int_{\gamma_r} f \to 0$ as $r \to 0$

Step 4 Study $\int_{\gamma_1} f = \int_{-R}^{-r} \frac{\ln x}{(x^2+1)^2}\, dx$

Let $t = -x$. Then $dt = -dx$. We get

$$\int_{\gamma_1} f = -\int_{R}^{r} \frac{\ln(-t)}{(t^2+4)^2}\, dt = \int_{r}^{R} \frac{\ln t + \pi i}{(t^2+4)^2}\, dt$$

$$= \int_{r}^{R} \frac{\ln t}{(t^2+4)^2}\, dt + \pi i \int_{r}^{R} \frac{1}{(t^2+4)^2}\, dt.$$

In the limit as $r \to 0$, $R \to \infty$, this becomes

$$\lim_{\substack{r \to 0 \\ R \to \infty}} \int_{\gamma_1} f = \underbrace{\int_0^\infty \frac{\ln x}{(x^2+4)^2}\, dx}_{= I} + \pi i \int_0^\infty \frac{1}{(x^2+4)^2}\, dx.$$

Step 4.5 We will show $\int_0^\infty \frac{1}{(x^2+4)^2}\, dx = \frac{\pi}{32}$.

Let $x = 2\tan\theta$, $0 \le \theta \le \frac{\pi}{2}$. Then $dx = 2\sec^2\theta\, d\theta$.

$(x^2+4)^2 = (4\tan^2\theta + 4)^2 = 16\sec^4\theta$.

$$\int_0^{\pi/2} \frac{2\sec^2\theta}{16\sec^4\theta}\, d\theta = \frac{1}{8}\int_0^{\pi/2} \cos^2\theta\, d\theta = \frac{1}{8}\int_0^{\pi/2} \frac{1+\cos(2\theta)}{2}\, d\theta$$

$$= \frac{1}{16}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/2} = \frac{1}{16}\cdot\frac{\pi}{2} = \frac{\pi}{32}.$$

Step 5 The Grand Finally!

$$\lim_{\substack{r\to 0\\ R\to\infty}} \int_\gamma f = \lim_{\substack{r\to 0\\ R\to\infty}} \left(\int_{\gamma_R} f + \int_{\gamma_1} f + \int_{\gamma_r} f + \int_{\gamma_2} f \right)$$

$$\frac{\pi(\ln 2-1)}{16} + \cancel{\frac{\pi^2}{32}i} = 0 + I + 0 + I + \pi i\left(\frac{\pi}{32}\right) = 2I + \cancel{\frac{\pi^2}{32}i}$$

Thus, $I = \dfrac{\pi(\ln 2-1)}{32} \approx -0.03$ as claimed!

Note: We should have confirmed $\int_0^\infty \frac{\ln x}{(x^2+4)^2}dx$ converges first. We do this here.

$$\int_1^\infty \frac{\ln x}{(x^2+4)^2}\,dx \le \int_1^\infty \frac{x}{(x^2+4)^2}\,dx \quad \text{clearly converges.}$$

To see that $\int_0^1 \frac{\ln x}{(x^2+4)^2}dx$ converges ($\ln x \to -\infty$ as $x\to 0^+$),

We notice that $\dfrac{1}{25} \le \dfrac{1}{(x^2+4)^2} \le \dfrac{1}{16}$ for $x\in[0,1]$

Thus $\dfrac{\ln x}{25} \ge \dfrac{\ln x}{(x^2+4)^2} \ge \dfrac{\ln x}{16}$ since $\ln x \le 0$ for $x\in(0,1]$

Since $\int_0^1 \ln x\,dx = x\ln x - x\Big|_0^1 = -\lim_{x\to 0}(x\ln x - x) = -1$ (use L'Hop on $\frac{\ln x}{\frac{1}{x}}$)

We have $-\dfrac{1}{25} \ge \displaystyle\int_0^1 \frac{\ln x}{(x^2+4)^2}dx \ge \dfrac{-1}{16}$

converges.