Example: Compute $\int_0^{\infty} \sin(x^2)\, dx$.

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> plot(sin(x^2),x=0..15);
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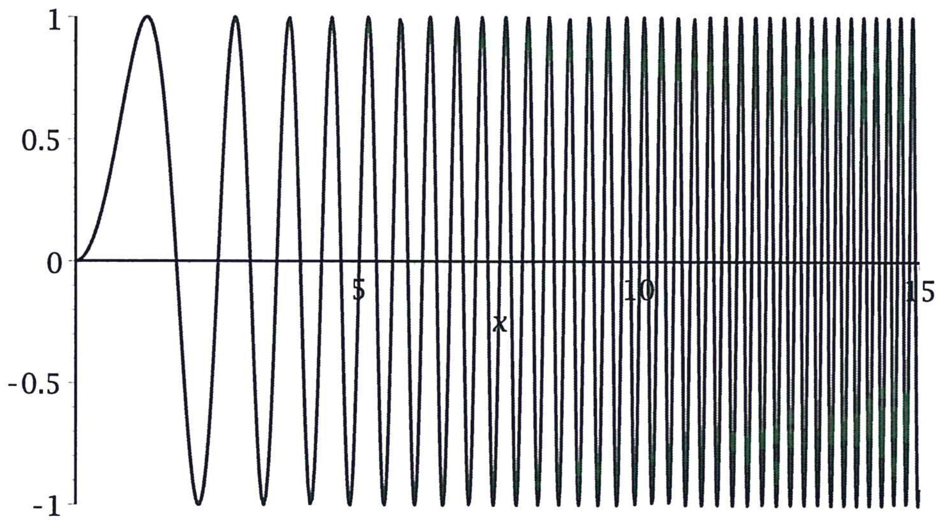
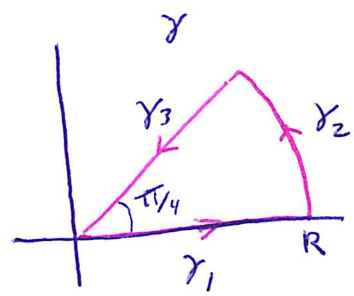


Solution (From Spiegel, 1964)



Consider the closed curve γ shown:

It can be broken up into 3 pieces:

$\gamma_1(t) = t \quad 0 \le t \le R.$

$\gamma_2(t) = Re^{it} \quad 0 \le t \le \frac{\pi}{4}$

$\gamma_3(t) = te^{i\frac{\pi}{4}} \quad 0 \le t \le R$ (going from R to 0).

We know $\int_{\gamma} e^{iz^2}\, dz = 0.$

Let $I_1 = \int_{\gamma_1} e^{iz^2}\, dz$, $I_2 = \int_{\gamma_2} e^{iz^2}\, dz$ and $I_3 = \int_{\gamma_3} e^{iz^2}\, dz$.

Then $I_1 + I_2 + I_3 = 0.$

We will show that $\lim\limits_{R\to\infty} I_2 = 0$ and $\lim\limits_{R\to\infty} I_3 = -\frac{\sqrt{\pi}}{2} e^{i\frac{\pi}{4}}$.

Therefore $\lim\limits_{R\to\infty} I_1 = \frac{\sqrt{\pi}}{2} e^{i\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}\left(\frac{1+i}{\sqrt{2}}\right) = \frac{1}{2}\sqrt{\frac{\pi}{2}} + \frac{1}{2}\sqrt{\frac{\pi}{2}}\, i$

But $\lim\limits_{R\to\infty} I_1 = \int_0^\infty \cos x^2\, dx + i\int_0^\infty \sin x^2\, dx$, since on the real axis $e^{iz^2} = e^{ix^2} = \cos x^2 + i\sin x^2$.

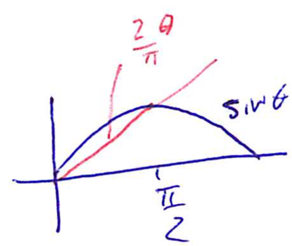
Thus, $\int_0^\infty \cos x^2\, dx = \int_0^\infty \sin x^2\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$.

Proof that $\lim\limits_{R\to\infty} I_2 = 0$.

$$\lim_{R\to\infty} \left| \int_0^{\frac{\pi}{4}} \left(e^{iR^2e^{i2t}}\right)\left(iRe^{it}\right) dt \right| \le \lim_{R\to\infty} R\int_0^{\frac{\pi}{4}} \left| e^{iR^2\cos 2t - R^2\sin 2t} \right| dt$$

$$= \lim_{R\to\infty} R\int_0^{\frac{\pi}{4}} e^{-R^2\sin 2t}\, dt = \lim_{R\to\infty} \frac{R}{2}\int_0^{\frac{\pi}{2}} e^{-R^2\sin\theta}\, d\theta = \star$$

Let $\theta = 2t$.
Then $dt = \frac{1}{2}d\theta$



Now for $\theta \in [0, 2\pi)$ we have $\frac{2}{\pi}\theta \le \sin\theta$.

Thus $e^{R^2\frac{2}{\pi}\theta} \le e^{R^2\sin\theta} \Rightarrow e^{-R^2\sin\theta} \le e^{-2R^2/\pi\,\theta}$.

Thus,

$$\star \le \lim_{R\to\infty} \frac{R}{2}\int_0^{\pi/2} e^{-\frac{2R^2\theta}{\pi}}\, d\theta = \lim_{R\to\infty} \frac{R}{2}\left(\frac{\pi}{-2R^2} e^{-\frac{2R^2\theta}{\pi}}\right)\Big|_0^{\pi/2} =$$

$$\lim_{R\to\infty} -\frac{\pi}{4R}\left(e^{-R^2} - 1\right) = \lim_{R\to\infty} \frac{\pi}{4R}\left(1 - e^{-R^2}\right) = 0.$$

Finally we show that $\lim_{R\to\infty} I_3 = -\frac{\sqrt{\pi}}{2} e^{i\frac{\pi}{4}}$.

Recall $I_3 = \int_R^0 \left(e^{it^2 e^{i\frac{\pi}{2}}}\right)\left(e^{i\frac{\pi}{4}}\right) dt = -e^{i\frac{\pi}{4}}\int_0^R e^{-t^2}\,dt.$

We just need to show $\int_0^R e^{-t^2}dt$ goes to $\frac{\sqrt{\pi}}{2}$.

$$\left(\int_{-\infty}^{\infty} e^{-t^2}dt\right)^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}dy\right) = \int_{-\infty}^{+\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy$$

$$= \int_0^{\infty}\int_0^{2\pi} e^{-r^2} r\,d\theta\,dr = 2\pi \lim_{R\to\infty}\int_0^R e^{-r^2} r\,dr = *$$

Let $u = r^2$. $du = 2r\,dr$. Thus

$$* = \pi \lim_{R\to\infty}\int_0^{R^2} e^{-u}du = \pi \lim_{R\to\infty}\left(-e^{-u}\right)\Big|_0^{R^2}$$

$$= \pi \lim_{R\to\infty}\left(-e^{-R^2} - (-1)\right) = \pi.$$

Thus $\int_{-\infty}^{\infty} e^{-t^2}dt = \sqrt{\pi}$ and $\int_0^{\infty} e^{-t^2}dt = \frac{\sqrt{\pi}}{2}$.

Thus $I_3 = -e^{\frac{i\pi}{4}}\cdot\frac{\sqrt{\pi}}{2}$ as claimed. $\square$