The <u>Mellin Integral Transform</u> is

$$\mathcal{M}(f(x))(a) = \int_0^\infty x^{a-1} f(x)\,dx.$$

You can read more about it at

http://mathworld.wolfram.com/MellinTransform.html

or http://en.wikipedia.org/wiki/Mellin_transform.html

We will compute the Mellin transform of $f(x) = \frac{1}{1+x^2}$ at $a = \frac{4}{3}$. This is Example 4.3.15 in the textbook. Proposition 4.3.16 generalizes ~~this~~ example. We will show that

$$\int_0^\infty \frac{\sqrt[3]{x}}{1+x^2}\,dx = \frac{\pi}{\sqrt{3}}.$$

First, we not that this improper integral does converge. $\int_0^1 \frac{\sqrt[3]{x}}{1+x^2}\,dx$ is not a problem, it clearly exists. Next,

$$\int_1^\infty \frac{x^{\frac{1}{3}}}{1+x^2}\,dx \le \int_1^\infty \frac{x^{\frac{1}{3}}}{x^2}\,dx = \int_1^\infty x^{-5/3}\,dx = -\frac{3}{2} x^{-\frac{2}{3}}\Big|_1^\infty = \frac{3}{2}.$$

Now, our plan is too "complexify" the problem and study

$$\int_\gamma g(z)\,dz, \text{ with } g(z) = \frac{\sqrt[3]{z}}{1+z^2} \text{ for some}$$

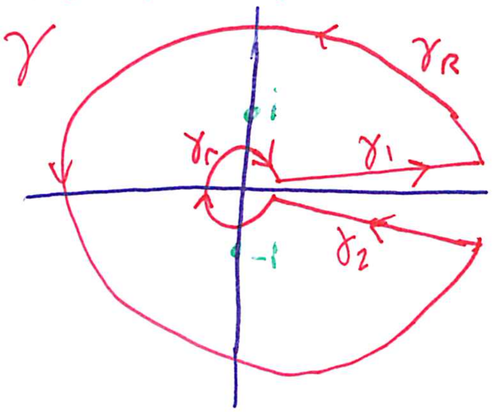
clever closed curve γ. But $z^{\frac{1}{3}} = e^{\frac{1}{3}\log z}$ is not analytic everywhere. We will need to use a <u>branch cut</u>.

This may seem weird, but we will cut out the origin and the positive real axis. γ will be the curve shown below.

$0 < r < 1$

$0 < \eta < \frac{\pi}{2}$

$R > 1$



$\gamma_R = Re^{it} \quad t \in [\eta, 2\pi - \eta]$

$\gamma_r = re^{it} \quad t \in [\eta, 2\pi - \eta]$

$\gamma_1 = te^{i\eta} \quad t \in [r, R]$

$\gamma_2 = te^{(2\pi-\eta)i} \quad t \in [r, R]$

with orientations as shown.

Note: We cannot reduce γ_2 to $te^{-\eta i}$, as I'll explain in class.

Step 1

$$\int_\gamma \frac{\sqrt[3]{z}}{1+z^2}\,dz = 2\pi i\left(\operatorname{Res}(g, i) + \operatorname{Res}(g, -i)\right).$$

There are simple poles at i and $-i$.

$$\operatorname{Res}(g, i) = \frac{\sqrt[3]{i}}{2i} = \frac{e^{\frac{\pi i}{6}}}{2i} = \frac{\frac{\sqrt{3}}{2} + i\frac{1}{2}}{2i} = \frac{1}{4} - \frac{\sqrt{3}}{4}i$$

$$\operatorname{Res}(g, -i) = \frac{\sqrt[3]{-i}}{2(-i)} = \frac{i}{-2i} = -\frac{1}{2}$$

Thus, $$\int_\gamma g(z)\,dz = 2\pi i\left(-\frac{1}{4} - \frac{\sqrt{3}}{4}i\right) = \pi\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = X.$$

Now $X = \int_{\gamma_1} g + \int_{\gamma_R} g + \int_{\gamma_2} g + \int_{\gamma_r} g$. Let $I = \int_{-\infty}^{\infty} g(x)\,dx$.

We will take a limit as $r \to 0$, $R \to \infty$, $\eta \to 0$. We will show $\int_{\gamma_R} g \to 0$ and $\int_{\gamma_r} g \to 0$. You might think $\int_{\gamma_1} g$ and $\int_{\gamma_2} g$ will cancel out in the limit. They won't. instead $\int_{\gamma_1} g \to I$ and $\int_{\gamma_2} g \to FI$, where F is a factor we can compute. Then $X = I + 0 + FI + 0 \Rightarrow I = \frac{X}{1+F}$, as we are done!

Step 2 $\int_{\gamma_R} g(z)\,dz \to 0$ as $R \to \infty$ (and $\eta \to 0$, $r \to 0$).

$$\left|\int_{\gamma_R} g(z)\,dz\right| = \left|\int_{\eta}^{2\pi-\eta} g(Re^{it})\,Rie^{it}\,dt\right| \le \int_{\eta}^{2\pi-\eta} |g(Re^{it})|\,R\,dt$$

$$\le R\int_0^{2\pi} |g(Re^{it})|\,dt \le R\cdot\frac{R^{1/3}}{R^2-1}\cdot 2\pi = 2\pi\frac{R^{4/3}}{R^2-1} \to 0 \text{ as } R\to\infty.$$

Step 3 $\int_{\gamma_r} g(z)\,dz \to 0$ as $r \to 0$ (and $\eta \to 0$, $R \to \infty$)

The steps are almost the same.

$$\left|\int_{\gamma_r} g(z)\,dz\right| = \left|\int_{2\pi-\eta}^{\eta} g(re^{it})\,rie^{it}\,dt\right| \le \frac{2\pi\, r^{4/3}}{1-r^2} \to 0 \text{ as } r\to\infty.$$

Step 4 We study $\int_{\gamma_1} g(z)\,dz$. Remember $\gamma_1 = te^{\eta i}$, $r \le t \le R$.

$$\int_{\gamma_1} g = \int_r^R \frac{t^{1/3}e^{\eta i/3}}{1+t^2e^{2\eta i}}\,e^{\eta i}\,dt = e^{\frac{4\eta i}{3}}\int_r^R \frac{t^{\frac{1}{3}}}{1+t^2e^{2\eta i}}\,dt$$

As $\eta\to 0$, $e^{\frac{4\eta i}{3}} \to 1$ and $e^{2\eta i}\to 1$. (Convergence is uniform.)

Taking limits as $r\to 0$ and $R\to\infty$ gives,

$$\lim \int_{\gamma_1} g = \int_0^\infty \frac{t^{1/3}}{1+t^2}\,dt = I.$$

Step 6   We study $\int_{\gamma_2} g(z)\,dz$. Now $\gamma_2 = te^{(2\pi-\eta)i}$ $r \le t \le R$, going

$$\int_{\gamma_2} g = \int_R^r \frac{t^{1/3} e^{\frac{(2\pi-\eta)i}{3}}}{1+t^2 e^{2(2\pi-\eta)i}} e^{(2\pi-\eta)i}\,dt$$

$$e^{\frac{4(2\pi-\eta)i}{3}} \int_R^r \frac{t^{\frac{1}{3}}}{1+t^2 e^{4\pi i - 2\eta i}}\,dt$$

$$= -e^{\frac{4(2\pi-\eta)i}{3}} \int_r^R \frac{t^{1/3}}{1+t^2 e^{-2\eta i}}\,dt.$$

In limit as $\eta \to 0$, we get $-e^{\frac{8\pi}{3}i} \int_r^R \frac{t^{1/3}}{1+t^2}\,dt$

As $r \to 0$ and $R \to \infty$, this becomes

$$-e^{\frac{8\pi}{3}i} I. \quad \text{Thus } F = -e^{\frac{8\pi}{3}i} = -e^{\frac{2\pi i}{3}}.$$

Step 7

$$I = \frac{X}{1+F} = \frac{\pi\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)}{1-\cos\left(\frac{2\pi}{3}\right) - i\sin\left(\frac{2\pi}{3}\right)}$$

$$= \frac{\pi}{2}\,\frac{\sqrt{3}-i}{1-\left(-\frac{1}{2}\right)-\frac{\sqrt{3}}{2}i} = \frac{\pi}{2}\,\frac{\sqrt{3}-i}{\frac{3}{2}-\frac{\sqrt{3}}{2}i}$$

$$= \pi\,\frac{\sqrt{3}-i}{3-\sqrt{3}i} = \frac{\pi}{\sqrt{3}}\,\frac{\sqrt{3}-i}{\sqrt{3}-i} = \frac{\pi}{\sqrt{3}}$$