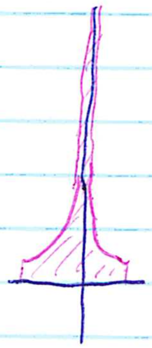
What happens if there is a pole on the real x-axis? Is there a way around it??

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_{-1}^{1} = (-1)-(--1) = -2, \text{ which is crazy.}$$



Area ≠ -2

The Fundamental Thm of Calculus does not ~~___~~ apply here because $\frac{1}{x^2}$ is not continuous over $[-1,1]$.

In this example we can write $\displaystyle\lim_{\varepsilon\to 0^+}\int_{\varepsilon}^{1}\frac{1}{x^2} = \infty$ and $\displaystyle\lim_{\varepsilon\to 0^-}\int_{-1}^{0}\frac{1}{x^2}\,dx = \infty$ and then say $\displaystyle\int_{-1}^{1}\frac{1}{x^2}\,dx = \infty$.

But what about $\displaystyle\int_{-1}^{1}\frac{1}{x^3}\,dx$? Now $\displaystyle\int_{0}^{1}\frac{1}{x^3}\,dx = \infty$ and $\displaystyle\int_{-1}^{0}\frac{1}{x^3}\,dx = -\infty$. So, can we say $\displaystyle\int_{-1}^{1}\frac{1}{x^3}\,dx = 0$?

Maybe. Limit as $\varepsilon\to 0$ of $\left[\displaystyle\int_{-1}^{-\varepsilon}\frac{1}{x^3}\,dx + \int_{\varepsilon}^{1}\frac{1}{x^3}\,dx\right] = 0$

This is called the Cauchy Principal Value of the integral.

However, when we apply this idea to the complex plane as before the "trick" we use only works for simple poles.

Lemma (4.3.10) Let $f(z)$ be analytic on an open deleted disk, $D(z_0,R)-\{z_0\}$ with a simple pole at z_0. Let γ_r be an arc of a circle $\gamma_r(t) = re^{it}$, $\alpha_0 \le t \le \alpha_0+\alpha$, $r<R$. Then

$$\lim_{r\to 0}\int_{\gamma_r} f(z)\,dz = \alpha\, i\, \mathrm{Res}(f, z_0)$$

We know this for $\alpha = 2\pi$. The proof is basically the same, see textbook.

Thm (Prop 4.3.11) Summary: Suppose $f(z)$ satisfies the conditions of Prop 4.3.6 or 4.3.9, but now may have a finite number of simple poles on real axis. Then,

$$P.V. \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum \text{residues of } f \text{ in upper half plane} + \pi i \sum \text{residues of } f \text{ on real axis}$$

Pf Combine Lemma with proofs of those Prop's.

Prop 4.3.12 does the same use lower half plane.

Your textbook does two examples:

Ex 4.3.13 $P.V. \int_{-\infty}^{\infty} \frac{x}{x^3+1}\,dx = \frac{\pi}{\sqrt{3}}$

Ex 4.3.14 $PV \int_{0}^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}$.