4.3 Evaluation of Definite Integrals

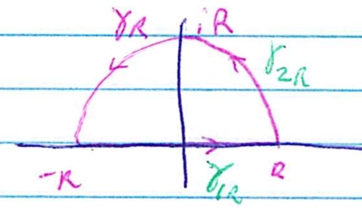
Ex $\displaystyle\int_0^\infty \frac{x^2}{x^6+1}\,dx.$

Sol. This can be done with Calc I methods. But we will develop a new method that will solve integrals that other methods cannot.

First, note that $\dfrac{x^2}{x^6+1} < \dfrac{x^2}{x^6} = \dfrac{1}{x^4}$ and $\displaystyle\int_1^\infty \frac{1}{x^4}\,dx = \frac{1}{3}.$

Clearly $\displaystyle\int_0^1 \frac{x^2}{x^6+1}\,dx$ exists. Thus, $\displaystyle\int_0^\infty \frac{x^2}{x^6+1}\,dx$ exists.

Let $R > 1$ and let γ_R be



We know $\displaystyle\int_{\gamma_R} \frac{z^2}{z^6+1}\,dz = \frac{\pi}{3}$ for all $R>1$.

Break up γ_R into two parts γ_{1R} on the real axis from $-R$ to R

and γ_{2R} the upper half circle radius R.

Now $\displaystyle\int_{\gamma_{1R}} \frac{z^2}{z^6+1}\,dz + \int_{\gamma_{2R}} \frac{z^2}{z^6+1}\,dz = \frac{\pi}{3}$

Claim $\displaystyle\lim_{R\to\infty} \int_{\gamma_{2R}} \frac{z^2}{z^6+1}\,dz = 0.$

Proof of Claim $|z^6+1| \geq R^6-1$ on γ_{2R}.

Thus, $\left|\dfrac{z^2}{z^6+1}\right| \leq \dfrac{R^2}{R^6-1}$.

Thus,

$$\left|\int_{\gamma_{2R}} \frac{z^2}{z^6+1}\,dz\right| = \int_{\gamma_{2R}} \left|\frac{z^2}{z^6+1}\right| |dz| \leq \frac{R^2}{R^6-1} \cdot \frac{2\pi R}{2} \to 0$$

as $R \to \infty$.

Now $\cancel{\int_{-\infty}^{\infty} \frac{x^2}{x^6+1}\,dx}$ $\int_{\gamma_{1R}} \dfrac{z^2}{z^6+1}\,dz = \displaystyle\int_{-R}^{R} \frac{x^2}{x^6+1}\,dx$

Since $\gamma_{1R} = t$ with $-R \leq t \leq R$. Let $t = x$.

Finally,

$$\int_{-\infty}^{\infty} \frac{x^2}{x^6+1}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{x^2}{x^6+1}\,dx = \lim_{R\to\infty} \frac{\pi}{3} = \frac{\pi}{3}.$$

$$\Rightarrow \int_0^{\infty} \frac{x^2}{x^6+1}\,dx = \frac{\pi}{6}$$

Calc I Method

Let $u = x^3$. Then $du = 3x^2\,dx$.

$$\int_0^{\infty} \frac{x^2}{x^6+1}\,dx = \frac{1}{3}\int_0^{\infty} \frac{1}{u^2+1}\,du = \frac{1}{3}\lim_{R\to\infty} \left(\arctan(R) - \arctan(0)\right)$$

$$= \frac{1}{3}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{6}$$

When does this trick work?

Thm (Prop 4.3.6)

(i) Let f be analytic on an open set containing the upper half plane except for a finite number of iso. sing. none of which lie on the real axis. Suppose there are real numbers $M \geq 0$, $p > 1$, $R_0 \geq 0$ s.t.

$$|f(z)| \leq \frac{M}{|z|^p} \quad \text{for } |z| \geq R_0, \ \operatorname{Im} z_0 \geq 0.$$

Then

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum \{\text{residues of } f \text{ in upper half plane}\}$$

(ii) A similar state holds for lower half plane (see textbook)

(iii) If $f = \frac{P}{Q}$ for polynomials P and Q with $\deg Q \geq 2 + \deg P$ and Q has no real zeros, then (i) and (ii) hold.

pf See textbook, but it is basically the same as our example.

Another Example

Compute $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\,dx$.

Sol 1 Conditions for prop 4.3.6. are clearly met.

Only one pole in ~~the~~ upper half plane, $z_0 = i$, order 2.

Let $g(z) = 1$, $h(z) = (z^2+1)^2$. $g(i) = 1 \neq 0$. $g'(i) = 0$

$h'(z) = 2(z^2+1)2z = 4z^3 + 4z$ $\quad h'(i) = 0$.

$h''(z) = 12z^2 + 4$ $\quad h''(i) = -8 \neq 0$. Use formula 6.

$h'''(z) = 24z$. $\quad h'''(i) = 24i$.

Thus $\displaystyle \text{Res}\left(\frac{1}{(z^2+1)^2}, i\right) = \frac{2g'(i)}{h''(i)} - \frac{2}{3}\frac{g(i)\,h'''(i)}{(h''(i))^2}$

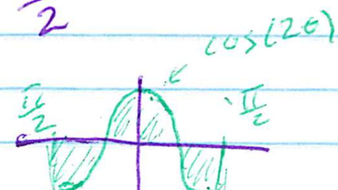
$$= 0 - \frac{2}{3}\,\frac{1\cdot 24i}{64} = -\frac{i}{4}.$$

Thus, $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\,dx = 2\pi i\left(-\frac{i}{4}\right) = \frac{\pi}{2}$.

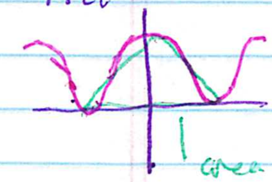
Sol 2 Let $x = \tan\theta$. $dx = \sec^2\theta\, d\theta$.

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\,dx = \int_{-\pi/2}^{\pi/2} \frac{\sec^2\theta\, d\theta}{\sec^4\theta} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\, d\theta = \frac{\pi}{2}$$



Trick:

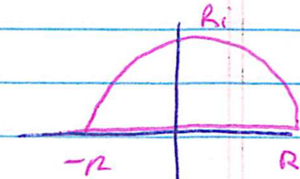


area $= \frac{1}{2}\pi\cdot 1 = \frac{\pi}{2}$

Example Evaluate $\int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2+2x+5}\, dx$.

Sol. Unfortunately, $|\sin \pi z|$ is not bounded as has exponential growth in the imaginary direction. The trick is to consider

$$\int_{\gamma} \frac{z e^{i\pi z}}{z^2+2z+5}\, dz \quad \text{where } \gamma =$$



for $R > |\text{roots of } z^2+2z+5|$.

Then we will extract the imaginary part along the real axis.

$z^2+2z+5=0$ at $z=-1\pm 2i$. Only $-1+2i$ is in upper half plane. It is a simple pole.

$$\operatorname{Res}\left(\frac{z e^{i\pi z}}{z^2+2z+5}, -1+2i\right) = \frac{(-1+2i)e^{i\pi(-1+2i)}}{2(-1+2i)+2} \quad \text{(by formula 4.)}$$

$$= -\left(\frac{1}{4}+\frac{i}{2}\right)e^{-2\pi}. \qquad \int_\gamma \frac{z e^{i\pi z}}{z^2+2z+5}\, dz = 2\pi i\left(\frac{1}{4}+\frac{i}{2}\right)e^{-2\pi} = \boxed{\frac{\pi}{2}(1-2i)e^{-2\pi}}$$

We cannot apply Prop 4.3.6. But a variant of it still works.

Let $f(z) = \frac{z}{z^2+2z+5}$. Let $\gamma_2 = Re^{it}$ $\quad 0 \le t \le \pi$.

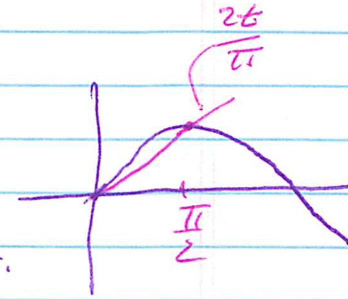
$$\left|\int_{\gamma_2} \frac{z e^{i\pi z}}{z^2+2z+5}\, dz\right| = \left|\int_0^{\pi} e^{i\pi \overset{Re^{it}}{\cancel{z}}} F(Re^{it})\, iRe^{it}\, dt\right|$$

$$\le \int_0^{\pi} \left| e^{i\pi Re^{it}} F(Re^{it})\, iRe^{it}\right| dt$$

$$= \int_0^{\pi} \left| e^{i\pi R\cos t - \pi R \sin t} F(Re^{it})\, iRe^{it}\right| dt$$

$$= \int_0^{\pi} e^{-\pi R \sin t} \left|F(Re^{it})\right| R \, dt$$

$$\le \int_0^{\pi} e^{-\pi R \sin t} \, dt \cdot R \cdot \frac{R}{R^2 - 2R - 5} = \star$$



For $0 \le t \le \frac{\pi}{2}$ we have $\sin t \ge \frac{2t}{\pi}$

Thus $e^{\pi R \sin t} \ge e^{2Rt} \Rightarrow e^{-\pi R \sin t} \le e^{-2Rt}$.

$$\int_0^{\pi} e^{-\pi R \sin t} \, dt = 2\int_0^{\pi/2} e^{-\pi R \sin t} \le 2\int_0^{\pi/2} e^{-2Rt} \, dt$$

$$= 2\left(-\frac{1}{2R} e^{-2t}\right)_0^{\pi/2} = -\frac{1}{R}\left(e^{-\pi} - 1\right) = \frac{1 - e^{-\pi}}{R}.$$

Thus, $$\star \le \frac{R}{R^2 - 2R - 5}\left(1 - e^{-\pi}\right)$$

The limit as $R \to \infty$ is zero.

Now we put the pieces together. See green box.

$$\frac{\pi}{2}(1 - 2i)e^{-2\pi} = \int_{\gamma} \frac{z e^{i\pi z}}{z^2 + 2z + 5} \, dz =$$

$$\cancel{\int_{-R}^{R} \frac{z e^{i\pi z}}{z^2 + 2z + 5}} \int_{-R}^{R} \frac{x e^{i\pi x}}{x^2 + 2x + 5} \, dx + \int_{\gamma_2} \frac{z e^{i\pi z}}{z^2 + 2z + 5} \, dz$$

Take limit as $R \to \infty$ and separate real and imaginary parts.

$$\frac{\pi}{2}(1-2i)e^{-2\pi} = \int_{-\infty}^{+\infty} \frac{x\cos \pi x}{x^2+2x+5}\,dx + i\int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2+2x+5}\,dx$$

Thus,

$$\int_{-\infty}^{\infty} \frac{x\cos \pi x}{x^2+2x+5}\,dx = \frac{\pi}{2}e^{-2\pi} \quad \text{and}$$

$$\int_{-\infty}^{\infty} \frac{x\sin \pi x}{x^2+2x+5}\,dx = -\pi e^{-2\pi}.$$

Note: This example is from Theory and Problems of Complex Variables by Spiegel, 1964.

There are <u>many</u> variations on these ideas.