Next up we study integrals of the form $\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$

where $R(x,y)$ is a rational function of two variables.

We will convert these into an contour integral around the unit circle in the complex plane of an auxiliary function:

$$f(z) = \frac{1}{iz} R\left(\frac{z + \frac{1}{z}}{2} , \frac{z - \frac{1}{z}}{2i} \right)$$

That looks a bit weird, but on the unit circle

$$z + \frac{1}{z} = e^{i\theta} + e^{-i\theta} = 2\cos\theta \quad \text{and}$$

$$z - \frac{1}{z} = e^{+i\theta} - e^{-i\theta} = 2i\sin\theta.$$

<u>Prop 4.3.2</u> Assume $R(x,y)$ has no sing. on $x^2+y^2=1$. Then

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta = 2\pi i \sum_{\text{inside unit circle}} \text{residues of } f(z)$$

<u>Proof</u>

$$\int_\gamma f(z)\, dz = 2\pi i \sum \text{res of } f \text{ inside } \gamma$$

$$\parallel \quad z = \gamma(\theta) = e^{i\theta}$$

$$\int_0^{2\pi} \frac{R(\cos\theta, \sin\theta)}{\cancel{ie^{i\theta}}} \cdot \cancel{ie^{i\theta}}\, d\theta.$$

Ex $\displaystyle\int_0^{2\pi} \frac{\cos(3\theta)}{5-4\cos\theta}\, d\theta = \frac{\pi}{12}$ (Spiegel, 1968)

Note $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$, so this is a rational function of $\cos\theta$.

Sol $\cos\theta = \dfrac{z+\frac{1}{z}}{2}$ and $\cos 3\theta = \dfrac{z^3+\frac{1}{z^3}}{2}$ on unit circle

$$\frac{\cos(3\theta)}{5-4\cos\theta} = \frac{(z^3+z^{-3})/2}{5-4(z+z^{-1})/2} = \frac{z^3+z^{-3}}{10-4(z+z^{-1})}$$

$$= \frac{z^6+1}{10z^3-4z^4-4z^2} = -\frac{1}{2z^2}\left(\frac{z^6+1}{2z^2-5z+2}\right) = -\frac{1}{2z^2}\left(\frac{z^6+1}{(2z-1)(z-2)}\right)$$

$$\int_0^{2\pi} \frac{\cos(3\theta)}{5-4\cos\theta}\, d\theta = -\frac{1}{2}\int_\gamma \frac{z^6+1}{z^2(2z-1)(z-2)} \frac{1}{iz}\, dz$$

$$= \frac{i}{2}\int_\gamma \frac{z^6+1}{z^3(2z-1)(z-2)}\, dz$$

There is a 3rd order pole at $z=0$, and a simple pole at $z=\frac{1}{2}$ inside unit circle γ.

Let $g(z) = z^6+1$, $h(z) = z^3(2z-1)(z-2)$

and $f(z) = \dfrac{g(z)}{h(z)}$.

$\mathrm{Res}\left(\frac{g}{h}, \frac{1}{2}\right)$: $g\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^6 + 1 = \frac{65}{64} \neq 0$

$h\left(\frac{1}{2}\right) = 0$

$h(z) = 2z^5 - 5z^4 + 2z^3$

$h'(z) = 10z^4 - \cancel{2}0z^3 + 6z^2$

$h'\left(\frac{1}{2}\right) = \frac{10}{16} - \frac{20}{8} + \frac{6}{4} = \frac{5}{8} - \frac{20}{8} + \frac{12}{8} = \frac{-3}{8} \neq 0$

Use formula 4.

$$\mathrm{Res}\left(f, \frac{1}{2}\right) = \frac{\frac{65}{64}}{-3/8} = -\frac{65}{64} \cdot \frac{8}{3} = -\frac{65}{24}.$$

$\mathrm{Res}(f, 0)$. Use formula 9 with $k = 3$.

$$\phi(z) = (z)^3 f(z) = \frac{z^6 + 1}{2z^2 - 5z + 2}$$

$\lim\limits_{z \to 0} \phi = \frac{1}{2} \neq 0$. Thus, $\mathrm{Res} = \lim\limits_{z \to 0} \frac{(\phi(z))''}{2!}$

$$\phi''(z) = \frac{6(8z^8 - 50z^7 + 106z^6 - 80z^5 + 20z^4 + 4z^2 - 10z + 7)}{(2z^2 - 5z + 2)^3}$$

(Via Maple)

$$\lim_{z \to 0} \phi''(z) = \frac{6 \cdot 7}{8} = \frac{21}{4}.$$

$\mathrm{Res}(f, 0) = \frac{21}{8}$. (Wolfram Alpha agrees)

Final answer is

$$\frac{1}{2} \cdot 2\pi i \left(\frac{21}{8} - \frac{65}{24}\right) = -\pi\left(\frac{63}{\cancel{2}4} - \frac{65}{\cancel{2}4}\right) = \frac{2\pi}{24} = \pi/12$$