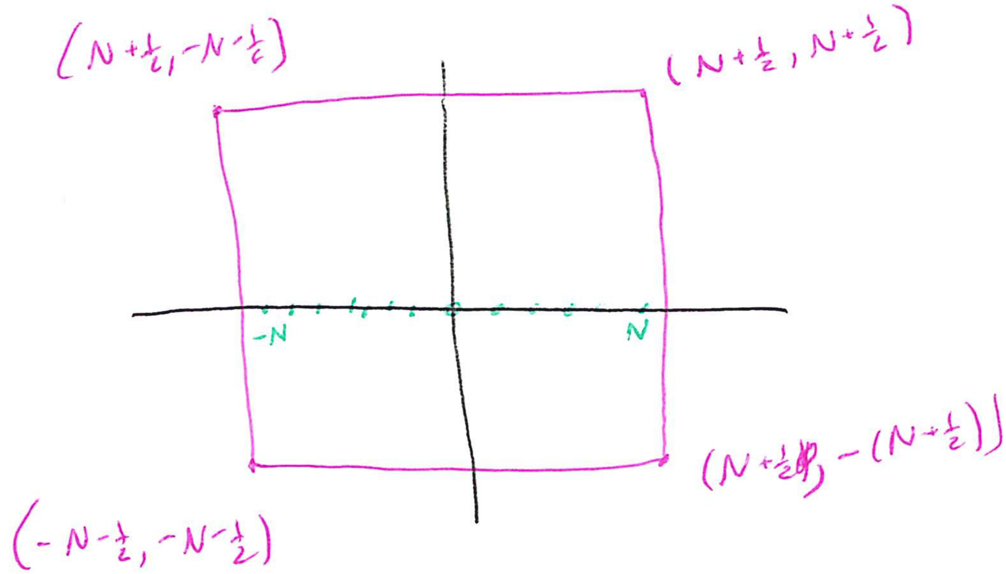
4.4

We are to develop a trick for computing certain infinite series. The series will be of the form

$$\sum_{n=-\infty}^{\infty} f(n)$$

where $f(z)$ is analytic except for a finite number of isolated singularities, not on real line. (Although we can sometimes get around this.)

We work with $\pi \cot(\pi z) f(z)$, which has simple poles at the integers. Let C_N be the closed curve that is a square in $\mathbb{C}$ with vertices $(\pm(N+\frac{1}{2}), \pm(N+\frac{1}{2}))$, $N = 1, 2, 3, \ldots$



Thm 4.4.1 Suppose $\int_{C_N} (\pi \cot \pi z) f(z)\, dz \to 0$ as $N \to \infty$.

Then

$$\lim_{N\to\infty} \sum_{n=-N}^{N} f(n) = - \sum \begin{pmatrix} \text{residues of } \pi\cot(\pi z) f(z) \\ \text{at the singularities of } f \end{pmatrix}$$

Pf $\int_{C_N} (\pi \cot \pi z) f(z)\, dz =$

$2\pi i \sum \left(\text{residues of } (\pi \cot \pi z) f(z) \text{ at } z = 0, \pm 1, \pm 2, \dots \pm N\right)$

$+ 2\pi i \sum \begin{pmatrix} \text{residues of } (\pi \cot \pi z) f(z) \text{ at the} \\ \text{sing. of } f \end{pmatrix}$

Once N is large enough that C_N contains all sing. of f.

You can check $\operatorname{Res}((\pi \cot \pi z) f(z), n) = f(n)$.

Now just take limits as $N \to \infty$.

$$\sum_{-\infty}^{\infty} f(n) = \lim_{N\to\infty} \sum \text{res of } (\pi \cot \pi z) f(z),\ z = 0, \pm 1, \dots \pm N$$

$$= - \sum \text{residues of } \pi \cot \pi z\, f(z) \text{ at sing of } f.$$

Since $\lim_{N\to\infty} \int_{C_N} \underline{\qquad}\, dz = 0$.

Prop. 4.4.2 gives condition under which Thm 4.4.1 works. Its proof is more involved.

Prop 4.4.2 Suppose f is analytic on $\mathbb{C}$ except for isol sing's. If $\exists\, R > 0,\ M > 0$ s.t. $|z f(z)| \le M$ when $|z| > R$, then $\int_{C_N} (\pi \cot \pi z) f(z)\, dz \to 0$ as $N \to \infty$.

Pf The sing of f are inside circle of radius R. (One can show they are finite)

$$\left| \frac{f(\frac{1}{z})}{z} \right| \le M \quad \text{for} \quad |z| < \frac{1}{R}$$

Therefore $\frac{f(\frac{1}{z})}{z}$ has only a removable sing at 0 and thus has a Taylor series

$$\frac{1}{z} f\left(\frac{1}{z}\right) = a_0 + a_1 z + a_2 z^2 + \cdots \qquad |z| < \frac{1}{R}$$

$$f\left(\frac{1}{z}\right) = a_0 z + a_1 z^2 + a_2 z^3 + \cdots \qquad |z| < \frac{1}{R}$$

$$f(z) = \frac{a_0}{z} + \frac{a_1}{z^2} + \frac{a_2}{z^3} + \cdots \qquad |z| > R$$

By the Residue Thm

$$\int_{C_N} \frac{\pi \cot \pi z}{z}\, dz = 2\pi i \sum \text{residues}$$

$$= 2\pi i \operatorname{Res}(0) + 2\pi i \sum_{\substack{n=-N \\ n \neq 0}}^{N} R(n)$$

We split off the Res at 0, as it is a pole of order two: $\frac{\pi \cos \pi z}{z \sin \pi z}$, while the others are simple poles.

But $\operatorname{Res}(0) = 0$ since $\dfrac{\pi \cot \pi z}{z}$ is an even function!

For the others $\operatorname{Res}(n) = \frac{1}{n}$; using formula 4.

$$\boxed{\begin{array}{ll} \dfrac{g(z) = \pi \cos \pi z}{h(z) = z \sin \pi z}, & g(n) = \pi \cos(\pi n) = (-1)^n \pi \neq 0. \\ & h(n) = n \sin(\pi n) = 0 \\ & h'(z) = \sin \pi z + \pi z \cos(\pi z) \\ & h'(n) = 0 + \pi n (-1)^n \\ \text{Thus, } \operatorname{Res}(n) = \dfrac{(-1)^n \pi}{\pi n (-1)^n} = \dfrac{1}{n}. & \end{array}}$$

Thus $\displaystyle\int_{C_N} \frac{\pi \cot \pi z}{z} = 0, \quad \left(-\frac{1}{n} - \frac{1}{n-1} - \dots - 1 + 0 + 1 + \frac{1}{2} + \dots + \frac{1}{n} \right)$

Thus, $\displaystyle\int_{C_N} \frac{\pi \cot \pi z}{} f(z)\, dz = \int_{C_N} \frac{\pi \cot \pi z}{} f(z)\, dz + 0$

$\displaystyle = \int_{C_N} \frac{\pi \cot \pi z}{} f(z)\, dz - a_0 \int \frac{\pi \cot \pi z}{z}\, dz$

$\displaystyle = \int_{C_N} \frac{\pi \cot \pi z}{} \left(f(z) - \frac{a_0}{z} \right) dz$ (weird!)

Now $f(z) - \frac{a_0}{z} = \frac{a_1}{z^2} + \frac{a_2}{z^3} + \frac{a_3}{z^4} + \cdots \quad |z| > R$

Remember $a_0 + a_1 z + a_2 z^2 + \cdots$ was analytic on $|z| < \frac{1}{R}$.

Then $a_1 + a_2 w + a_3 w^2 + \cdots$ is analytic on $|w| < \frac{1}{R}$.

Let $R' > R$. Then the closed Disk $|z| \leq \frac{1}{R'}$ is inside $D(0, \frac{1}{R})$

Then $a_1 + a_2 w + a_3 w^2 + \cdots$ is bdd on the compact set $|z| \leq \frac{1}{R'}$.

Let M' be such a bd. Thus

$$z \geq R' \Rightarrow \left| f(z) - \frac{a_0}{z} \right| \leq \frac{M'}{|z|^2}$$

Let $N \geq R'$. Then $|z| \geq R'$ on all of C_N.

Thus,
$$\left| \int_{C_N} \pi \cot(\pi z) \left[f(z) - \frac{a_0}{z} \right] dz \right|$$
$$\leq \pi \frac{M'}{(N+\frac{1}{2})^2} \cdot \underbrace{8\left(N+\tfrac{1}{2}\right)}_{\text{circumference of } C_N} \max_{z \in C_N} |\cot \pi z|$$

Extra Credit: Show that

$$\max_{z \in C_N} |\cot \pi z| = \frac{e^{2\pi(N+1/2)} + 1}{e^{2\pi(N+1/2)} - 1}.$$

Since $\lim_{N\to\infty} \dfrac{e^{2\pi(N+\frac{1}{2})}+1}{e^{2\pi(N+\frac{1}{2})}-1} = 1$, for N large

enough we can assume $\max|\cot \pi z| < 2$.

Now we have

$$\left| \int_{C_N} \pi \cot \pi z \, f(z)\, dz \right| \leq \frac{16\pi M'}{N+\frac{1}{2}} \to 0 \text{ as } N \to \infty.$$